Siberian Mathematical Journal, Vol. 47, *No.* 2, *pp.* 341–354, 2006

TOPOLOGICAL DIMENSIONS FOR u-GROUPS

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UDC 512.5

Abstract: We study some problems connected with algebraic geometry over a free metabelian group. We introduce the notions of topological dimensions which are based on the lengths of chains of irreducible closed sets, and study these dimensions.

Keywords: algebraic dimension, metabelian group, topological dimension

1. Introduction

The article is a part of a project of establishing algebraic geometry over a free metabelian group.

Let $G[X]$ be the free product of a given group G and the free group with basis $X = \{x_1, \dots, x_n\}$. The group $G[X]$ plays the role of the ring of polynomials in algebraic geometry over G. In accordance with [1], the set of solutions to some system of equations over $G[X]$ is called an *algebraic subset of the affine space* G^n. We endow G^n with the Zariski topology: the algebraic subsets in G^n are taken as a subbasis of the system of closed sets. Dual to the category of algebraic sets is the category of coordinate groups. If B is an algebraic set then the quotient group $G[X]$ by the annihilator of B is called the *coordinate group* of B.

The dimension of an algebraic set B is defined in a standard way; namely, as the number n such that B admits a chain of pairwise distinct irreducible closed sets:

$$B = B_0 \supset B_1 \supset \cdots \supset B_n,$$

and there is no chain with more terms. To a strictly decreasing chain of irreducible algebraic sets there corresponds a chain of proper epimorphisms (with nonidentity kernels) of the coordinate groups.

A group G is called a *u-group* if G enjoys the universal theory of a free metabelian group of rank ≥ 2. Every coordinate group is known to be a u-group [2, 3]. Therefore, it is interesting to study the lengths of chains of epimorphisms for u-groups. Depending on whether the chain contains abelian groups or not, we define two topological dimensions for u-groups. Our article is devoted to studying these dimensions.

Theorem 1 of this article calculates the topological dimension for the group $M(T_n, A_m)$ isomorphic to the discrete wreath product of free abelian groups of ranks n and m. Modifying the proof of Theorem 1, we find the nonabelian topological dimension of $M(T_n, A_m)$. We introduce the class $U_\searrow$ of splittable u-groups and prove that, for every nonabelian u-group G, there exists an embedding in the splittable envelope $G_{\text{split}} \in U_\searrow$. It turns out that the topological dimensions of G and G_{split} are closely connected. Moreover, the nonabelian topological dimensions of these groups coincide (Theorem 2). This enables us to calculate the topological dimensions of groups by considering their splittable envelopes. Along these lines, we find the topological dimensions of a free metabelian group (Theorems 3 and 4).

2. Splittable u-Groups

2.1. u-Groups. Suppose that G is a metabelian group, i.e., G has an abelian normal subgroup M such that $\overline{G} = G/M$ is an abelian group. The elements of G act on M by conjugation: $m^g = g^{-1}mg$,

Both authors were supported by the Russian Foundation for Basic Research (Grant 05.01.00292). Moreover, the second author was supported by the Scientific Program "Basic Research in Higher School: Universities of Russia" of the Ministry for Education of the Russian Federation (Grant UR.04.01.031).

Omsk; Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 47, No. 2, pp. 414–430, March–April, 2006. Original article submitted February 7, 2005.

$m \in M$, $g \in G$. Using this action, we endow M with the structure of a right $\mathbf{Z}\overline{G}$-module, where $\mathbf{Z}\overline{G}$ is the integral group ring of $\overline{G}$.

Denote by $\operatorname{Fit}(G)$ the Fitting radical of G, i.e., the subgroup generated by all nilpotent normal subgroups of G.

Definition. A torsion-free metabelian group G is called a *u-group* if G meets the following conditions:

(1) $\operatorname{Fit}(G)$ is an abelian group;
(2) $A = G/\operatorname{Fit}(G)$ is a torsion-free abelian group;
(3) $\operatorname{Fit}(G)$ is torsion-free as a $\mathbf{Z}A$-module.

The class of u-groups can be defined by universal axioms [3, 4]. Necessary information on u-groups and their relationship with algebraic geometry over groups can be found in [2–4].

The definition implies that a nonabelian u-group has trivial center.

In what follows, we refer to

Proposition 1 [2]. *Let G be a nonabelian u-group and let N be an isolated ideal in $\operatorname{Fit}(G)$. Then the quotient group G/N is a u-group.*

Some invariants $\alpha(G)$ and $\beta(G)$ were defined for every u-group G in [2]. Recall that $\alpha(G)$ is equal to the rank of the free abelian group $A = G/\operatorname{Fit}(G)$. Let n be the minimal rank of a free $\mathbf{Z}A$-module that includes $\operatorname{Fit}(G)$. Then $\beta(G) = n$. Equivalently, we can define $\beta(G)$ as the maximal cardinality of a system of elements in $\operatorname{Fit}(G)$ linearly independent over $\mathbf{Z}A$.

The following proposition collects some assertions in [2]:

Proposition 2. *If G_1 and G_2 are finitely generated u-groups and $\varphi : G_1 \to G_2$ is an epimorphism then*

(1) *$\alpha(G_2) \le \alpha(G_1)$;*
(2) *if $\alpha(G_2) = \alpha(G_1)$ then $\beta(G_2) \le \beta(G_1)$; if $\ker\varphi \ne 1$ and $\alpha(G_2) = \alpha(G_1)$ then $\beta(G_2) < \beta(G_1)$;*
(3) *if G is a nonabelian u-group admitting a system of n generators then $\beta(G) \le n - 1$.*

2.2. The class of groups $U_{\rightthreetimes}$. Denote by $U_{\rightthreetimes}$ the class of nonabelian u-groups G with the radical $\operatorname{Fit}(G)$ splittable in G. We call these groups *$u_{\rightthreetimes}$-groups.*

Let A be a group and let T be a $\mathbf{Z}A$-module. Denote by $M(T, A)$ the group of matrices

$$M(T, A) = \left\{ \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix},\ a \in A,\ t \in T \right\}.$$

Identify the group A with the matrix group $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the module T, with the module $\begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$.

If T is torsion-free then the Fitting radical of $M(T, A)$ coincides with the subgroup $\widetilde{T}$ and the Fitting quotient group is isomorphic to A.

Lemma 1. *Suppose that $G = M(T, A)$ and $\overline{G} = M(\overline{T}, \overline{A})$ and let $\varphi : G \to \overline{G}$ be an epimorphism. If G and $\overline{G}$ are u-groups then*

(1) *$\varphi(T) = \overline{T}$,*
(2) *$\overline{G} \cong M(\overline{T}, \varphi(A))$.*

Proof. (1) Since the epimorphism φ takes the Fitting radical into the Fitting radical, $\varphi(T) \le \overline{T}$. Suppose that $\bar\tau \in \overline{T}$ and $g = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ is the preimage of $\bar\tau$ in G. Then the image of the element $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is in $\overline{T}$. The element $\varphi(a)$ commutes with the subgroup $\varphi(A)$. On the other hand, $\varphi(a)$ belongs to $\overline{T}$ and, hence, commutes with $\varphi(T)$. Therefore, $\varphi(a) = 1$ and $\bar\tau$ is the image of $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

(2) Since $\varphi(T) = \overline{T}$, it suffices to prove that $\overline{T} \cap \varphi(A) = 1$. Assume that $\bar{t} \in \overline{T}$ and $\bar{t} = \varphi(a)$, $a \in A$. Then

$$[\varphi(a), \varphi(A)] = [\varphi(a), \overline{T}] = 1,$$

i.e., $\varphi(a) = 1$. The lemma is proven.

Proposition 3. *The following are equivalent for a nonabelian u-group G:*

(1) *G is a $u_{\leftthreetimes}$-group;*

(2) *$G \cong M(T, A)$ for some torsion-free abelian group A and some torsion-free $\mathbf{Z}A$-module T;*

(3) *G is a quotient of $M(F, A)$, where A is a torsion-free abelian group and F is a free $\mathbf{Z}A$-module.*

Proof. (1) $\Rightarrow$ (2). Put $T = \mathrm{Fit}(G), A = G/T$. Since the subgroup T is splittable in G, it follows that $G \cong M(T, A)$.

(2) $\Rightarrow$ (3). Suppose that F is a free $\mathbf{Z}A$-module and $F/N \cong T$. Then G is a homomorphic image of $M(F, A)$ and the kernel of the homomorphism coincides with the subgroup $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$.

(3) $\Rightarrow$ (1). Assume that $\varphi : M(F, A) \to G$ and $R = \ker \varphi$. Prove that

$$R = \begin{pmatrix} A_1 & 0 \\ R_1 & 1 \end{pmatrix}$$

for some $\mathbf{Z}A$-submodule R_1 in F and some abelian subgroup A_1 in A. Since G is a u-group, this will finish the proof.

Let $g = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ be an element in R. Prove that $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ also belong to R.

Indeed, there exists an element c in F that does not belong to R. Otherwise, G is abelian. Since c belongs to the Fitting radical of $M(F, A)$, its image $\varphi(c)$ belongs to $\mathrm{Fit}(G)$ and $\varphi(c) \neq 1$. The commutator $[c, g] = c^{a-1}$ lies in R. Therefore,

$$\varphi(c)^{\varphi(a)-1} = 1.$$

Since $\mathrm{Fit}(G)$ is torsion-free, $\varphi(a) = 1$. Hence, $a, t \in R$. The proposition is proven.

Since R is a normal subgroup in $M(F, A)$, it follows that $R_1 \geq F(A_1 - 1)$ and, thus,

$$G \cong \begin{pmatrix} A/A_1 & 0 \\ F/R_1 & 1 \end{pmatrix}.$$

We will need the following lemma which bounds $\beta(G)$ for finitely generated groups in $U_{\leftthreetimes}$.

Lemma 2. *Assume that a u-group G is a homomorphic image of the group $M(F, A)$, where F is a free $\mathbf{Z}A$-module of rank n. Then $\beta(G) \leq n$.*

Proof. Lemma 1 and Proposition 3 imply that $G = M(T, B)$, where the $\mathbf{Z}B$-module T is a homomorphic image of the $\mathbf{Z}A$-module F; moreover, the module homomorphism agrees with the group homomorphism $A \to B$. Consequently, T is generated by at most n elements. Since $\beta(G)$ is the cardinality of a maximal linear independent system of elements in T, it follows that $\beta(G) \leq n$.

2.3. Splittable envelopes and standard embeddings. The following proposition makes it possible to embed an arbitrary u-group in a group of class $U_{\leftthreetimes}$ having the same α and β as the initial group.

Proposition 4. *Suppose that G is a nonabelian u-group; $T = \mathrm{Fit}(G)$; $\overline{G} = G/T$; $\bar{g}$ is the image of an element $g \in G$ under the natural homomorphism $G \to \overline{G}$; $n \geq 1$; $h_1, \dots, h_n$ are fixed elements in G. Assume that $\bar{h}_1 \neq 1$ and $\bar{h}_1$ is different from $\bar{h}_2, \dots, \bar{h}_n$. Then the mapping*

$$g \mapsto \begin{pmatrix} \bar{g} & 0 \\ \prod_{i=1}^{n} [g, h_i] & 1 \end{pmatrix}$$

is an embedding of G in the matrix group $M(T,\overline{G})$. Moreover, if δ is the element $\bar{h}_1+\dots+\bar{h}_n-n$ in $\mathbf{Z}\overline{G}$ then

$$T^\alpha=\begin{pmatrix}1 & 0\\ T\cdot\delta & 1\end{pmatrix}.$$

PROOF. For $j=1,2$, we have

$$g_j^\alpha=\begin{pmatrix}\bar{g}_j & 0\\ \prod_{i=1}^{n}[g_j,h_i] & 1\end{pmatrix}.$$

Then

$$(g_1g_2)^\alpha=\begin{pmatrix}\bar{g}_1\bar{g}_2 & 0\\ \prod_{i=1}^{n}[g_1g_2,h_i] & 1\end{pmatrix}=\begin{pmatrix}\bar{g}_1\bar{g}_2 & 0\\ \prod_{i=1}^{n}[g_1,h_i]^{g_2}[g_2,h_i] & 1\end{pmatrix}.$$

Since, in the module notation,

$$[g_1,h_i]^{g_2}[g_2,h_i]=[g_1,h_i]\cdot\bar{g}_2+[g_2,h_i],$$

it follows that $(g_1g_2)^\alpha=g_1^\alpha g_2^\alpha$.

Prove that $\ker\varphi=1$. Take $g\in\ker\varphi$. Then g belongs to T. Consequently,

$$\prod_{i=1}^{n}[g,h_i]=g^{\bar{h}_1+\dots+\bar{h}_n-n}.$$

By assumption, the element $\bar{h}_1+\dots+\bar{h}_n-n$ is nonzero. Since the $\mathbf{Z}\overline{G}$-module T is torsion-free, $g=1$. The proposition is proven.

Let G be a u-group. We call a group H of class $U_\curlywedge$ a *splittable envelope* for G if the groups $\overline{H}=H/\operatorname{Fit}(H)$ and $\overline{G}=G/\operatorname{Fit}(G)$ are isomorphic and the $\mathbf{Z}\overline{G}$-module $\operatorname{Fit}(G)$ is isomorphic to the $\mathbf{Z}\overline{H}$-module $\operatorname{Fit}(H)$. If A is an abelian group then a splittable envelope of A is by definition A. Proposition 4 enables us to construct a splittable envelope for an arbitrary u-group.

We call the group $M(T,\overline{G})$ and the embedding $G\xrightarrow{\alpha}M(T,\overline{G})$ constructed in Proposition 4 the *standard splittable envelope* and *standard embedding* corresponding to $h_1,\dots,h_n$. Denote them by $G_{\mathrm{split}}(h_1,\dots,h_n)$ and $\alpha(h_1,\dots,h_n)$. If the elements $h_1,\dots,h_n$ are predefined then we write G_{split} and α.

It is easy to prove

Proposition 5 (Occurrence Criterion). *Let $\alpha: G\to G_{\mathrm{split}}$ be the standard embedding of a nonabelian u-group G in the standard splittable envelope corresponding to $g_1,\dots,g_n$. Put $T=\operatorname{Fit}(G)$, $\overline{G}=G/T$, $\delta=\bar{g}_1+\dots+\bar{g}_n-n$. An element $\begin{pmatrix}a & 0\\ t & 1\end{pmatrix}$, $a\in\overline{G}$, $t\in T$, of G_{split} lies in G^α if and only if*

(1) *there exists $g\in G$ such that $\bar{g}=a$;*

(2) *the element $t-\prod_{i=1}^{n}[g,g_i]$ of the $\mathbf{Z}\overline{G}$-module T divides by δ.*

PROOF immediately ensues from the fact that $\prod_{i=1}^{n}[g,g_i]=g^\delta$ for $g\in T$.

Proposition 6. *Let $G_1\xrightarrow{\varphi}G_2$ be an epimorphism of nonabelian u-groups and let g be an element of G_1 with the property that g^φ does not belong to the subgroup $\operatorname{Fit}(G_2)$, $G_{1,\mathrm{split}}=G_{1,\mathrm{split}}(g)$, $G_{2,\mathrm{split}}=G_{2,\mathrm{split}}(g^\varphi)$. Then there exists a homomorphism*

$$\psi: G_{1,\mathrm{split}}\to G_{2,\mathrm{split}}$$

such that the diagram

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"'] \arrow[d, "\alpha_1(g)"] & G_2 \arrow[d, "\alpha_2(g^\varphi)"] \\ G_{1,\mathrm{split}} \arrow[r, "\psi"'] & G_{2,\mathrm{split}} \end{tikzcd}$$

commutes.

PROOF. Suppose that $T_i = \mathrm{Fit}(G_i)$, $\overline{G}_i = G_i/T_i$, $f \in G_1$, $t \in T_1$. Define a mapping $\psi : M(T_1, \overline{G}_1) \to M(T_2, \overline{G}_2)$ by the rule

$$\begin{pmatrix} \bar{f} & 0 \\ t & 1 \end{pmatrix} \overset{\psi}{\mapsto} \begin{pmatrix} \bar{f}^{\overline{\varphi}} & 0 \\ t^{\varphi} & 1 \end{pmatrix},$$

where $\overline{\varphi} : \overline{G}_1 \to \overline{G}_2$ is the induced homomorphism.

It is easy to check that ψ is a homomorphism and the diagram commutes. The assertion is proven.

We can show that the so-constructed homomorphism is not necessarily an epimorphism.

2.4. Localization of groups of class $U_{\leftthreetimes}$. Let $G = M(T, A)$ be a group of class $U_{\leftthreetimes}$, $R = \mathbf{Z}A$. Choose a multiplicatively closed set S with unity in the ring R. Standardly construct the quotient ring $S^{-1}R$ with respect to S and the $S^{-1}R$-module $S^{-1}T$. Note that R is embedded in $S^{-1}R$ and the R-module T is embedded in the $S^{-1}R$-module $S^{-1}T$. Moreover, we obtain an embedding of G into the group $G_s = M(S^{-1}T, A)$ of class $U_{\leftthreetimes}$, defined as follows:

$$\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \mapsto \begin{pmatrix} a/1 & 0 \\ t/1 & 1 \end{pmatrix}.$$

DEFINITION. The group G_s is called the *localization* of the $u_{\leftthreetimes}$-group G with respect to S.

Proposition 7. *Suppose that $G = M(T, A)$ and $\overline{G} = M(\overline{T}, \overline{A})$ are groups of class $U_{\leftthreetimes}$ and $\varphi : G \to \overline{G}$ is an epimorphism and $\varphi(A) = \overline{A}$. Put $R = \mathbf{Z}A$, $\overline{R} = \mathbf{Z}\overline{A}$ and let $\psi : R \to \overline{R}$ be the ring homomorphism induced mapping $\varphi : A \to \overline{A}$. Let S be a multiplicatively closed subset in R and let $\overline{S}$ be its image in $\overline{R}$. If the intersection $S \cap \ker \psi$ is empty then the epimorphism φ can be extended to a group epimorphism*

$$\varphi_s : G_s \to \overline{G}_{\bar{s}}$$

of the localizations of these groups with respect to S and $\overline{S}$.

PROOF. Consider the mapping of groups $\varphi_s : G_s \to \overline{G}_{\bar{s}}$ defined as

$$\varphi_s : \begin{pmatrix} a & 0 \\ t/s & 1 \end{pmatrix} \mapsto \begin{pmatrix} a^{\varphi} & 0 \\ t^{\varphi}/s^{\psi} & 1 \end{pmatrix}.$$

If $t_1/s_1 = t_2/s_2$ then

$$\begin{pmatrix} a & 0 \\ t_1/s_1 & 1 \end{pmatrix}^{\varphi_s} = \begin{pmatrix} a & 0 \\ t_2/s_2 & 1 \end{pmatrix}^{\varphi_s},$$

i.e., φ_s is well defined. Check that φ_s is a homomorphism. Suppose that

$$\begin{pmatrix} a_i & 0 \\ t_i/s_i & 1 \end{pmatrix}^{\varphi_s} = \begin{pmatrix} a_i^{\varphi} & 0 \\ t_i^{\varphi}/s_i^{\psi} & 1 \end{pmatrix},$$

$i = 1, 2$. We have

$$\begin{pmatrix} a_1 & 0 \\ t_1/s_1 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ t_2/s_2 & 1 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & 0 \\ (t_1 \cdot a_2 s_2 + t_2 s_1)/s_1 s_2 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} a_1 a_2 & 0 \\ (t_1 \cdot a_2 s_2 + t_2 s_1)/s_1 s_2 & 1 \end{pmatrix}^{\varphi_s} = \begin{pmatrix} a_1^{\varphi} a_2^{\varphi} & 0 \\ (t_1 \cdot a_2 s_2 + t_2 s_1)^{\varphi}/s_1^{\psi} s_2^{\psi} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_1^{\varphi} a_2^{\varphi} & 0 \\ (t_1^{\varphi} \cdot (a_2 s_2)^{\psi} + t_2^{\varphi} s_1^{\psi})/s_1^{\psi} s_2^{\psi} & 1 \end{pmatrix} = \begin{pmatrix} a_1^{\varphi} & 0 \\ t_1^{\varphi}/s_1^{\psi} & 1 \end{pmatrix} \begin{pmatrix} a_2^{\varphi} & 0 \\ t_2^{\varphi}/s_2^{\psi} & 1 \end{pmatrix}.$$

The proposition is proven.

Proposition 8. *Let G_s be the localization of a $u_\curlywedge$-group $G = M(T, A)$ with respect to S. If S does not contain the zero element then $\beta(G) = \beta(G_s)$.*

The following proposition is useful in calculating the topological dimensions of splittable u-groups:

Proposition 9. *Let $\varphi : G \to \overline{G}$ be an epimorphism of finitely generated groups of class $U_\curlywedge$. If $\alpha(\overline{G}) = \alpha(G) - 1$ then $\beta(\overline{G}) \leq \beta(G)$.*

Proof. Suppose that $G = M(T, A)$, $\overline{G} = M(\overline{T}, \overline{A})$, where A and $\overline{A}$ are free abelian groups of finite ranks, $R = \mathbf{Z}A, \overline{R} = \mathbf{Z}\overline{A}$, and T and $\overline{T}$ are finitely generated torsion-free modules over R and $\overline{R}$ respectively.

Denote by φ_A the restriction of φ to the subgroup $A = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. By Lemma 1, we may assume that $\varphi_A(A) = \overline{A}$. Again by Lemma 1, $\varphi(T) = \begin{pmatrix} 1 & 0 \\ \overline{T} & 1 \end{pmatrix} = \overline{T}$.

Let $a_1, \dots, a_m$ be a basis of A, $\varphi_A(a_j) = \bar{a}_j$, $j = 2, \dots, m$, $\varphi_A(a_1) = 1$, and the elements $\bar{a}_2, \dots, \bar{a}_m$ form a basis of $\overline{A}$.

Take the set of nonzero elements of the integral group ring $\mathbf{Z}\langle a_2, \dots, a_m\rangle$ as a multiplicative set S.

Extend the homomorphism $\varphi_A : A \to \overline{A}$ to a ring homomorphism $\widetilde{\varphi}_A : R \to \overline{R}$. Since $\ker \widetilde{\varphi}_A \cap S$ is empty, by Proposition 7 the epimorphism φ can be extended to an epimorphism of the corresponding localizations:

$$\varphi_S : G_S \to \overline{G}_{\overline{S}},$$

where $\overline{S} = \varphi_A(S)$. Moreover, $\varphi_S(S^{-1}T) = \overline{S}^{-1}\overline{T}$. By Proposition 8,

$$\beta(G) = \beta(G_S), \quad \beta(\overline{G}) = \beta(\overline{G}_{\overline{S}}).$$

The quotient ring $\overline{S}^{-1}\overline{R}$ is a field. Denote it by P. Then $S^{-1}R = P\langle a_1\rangle$ is the ring of Laurent polynomials in a_1 over the field P. Hence, $S^{-1}R$ is a principal ideal domain. In this case, the torsion-free module $S^{-1}T$ is free over $S^{-1}R$.

Denote by $\beta(T)$ the maximal number of linearly independent elements in the R-module T. We infer

$$\beta(\overline{G}) = \beta(\overline{T}) = \beta(\overline{S}^{-1}\overline{T}) = \beta(\varphi_S(S^{-1}T)).$$

But $S^{-1}T$ is a free $S^{-1}R$-module. Therefore,

$$\beta(\varphi_S(S^{-1}T)) \leq \beta(S^{-1}T) = \beta(G).$$

The proposition is proven.

3. Topological Dimension for u-Groups

Definition. Let G be a finitely generated u-group. Consider the sequence of u-groups G_i and epimorphisms φ_j with nontrivial kernels $\ker(\varphi_j)$:

$$G = G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{l-1}} G_{l-1} \xrightarrow{\varphi_l} 1.$$

Call such a sequence a *u-sequence of length $l = l(G)$*.

From Proposition 3 it follows that, for a $u_\curlywedge$-group G, all nonabelian terms of the sequence are $u_\curlywedge$-groups.

Definition. The maximal length $l(G)$ of all u-sequences for a finitely generated u-group G is called its *topological dimension* and denoted by $\mathrm{tdim}(G)$. If we restrict exposition only to the u-sequences whose all terms are nonabelian u-groups then we arrive at the notions of a *nonabelian u-sequence* and the *nonabelian topological dimension* which we denote by $\mathrm{tdim}_0(G)$.

We put the topological dimensions of the trivial group to be zero.

From the standpoint of applications to algebraic geometry over a group G, the nonabelian topological dimension is more substantial.

Denote by $r_0(G)$ the number of infinite cyclic groups in the decomposition of an abelian group $G/[G,G]$ into a product of cyclic groups. It is easy to establish the following inequality:

$$\mathrm{tdim}(G) \leq \mathrm{tdim}_0(G) + r_0(G).$$

Lemma 3. *Let A be a free abelian group with basis $a_1, \ldots, a_m$ and let T be a free $\mathbf{Z}A$-module with basis $t_1, \ldots, t_n$, $n \geq 2$. For $i = 1, \ldots, n-1$, we put*

$$\tau_i = t_i(1 - a_1) + t_{i+1}(1 - a_2).$$

Let T_i be the submodule in T generated by $\tau_1, \ldots, \tau_i$. Then T_i are isolated submodules in T.

Proof. Assume that some $t \in T$ and $0 \neq \gamma \in \mathbf{Z}A$ meet the conditions $t \notin T_i$ but $t\gamma \in T_i$. We may assume that γ is a nondecomposable element of the integral domain $\mathbf{Z}A$. Suppose that

$$t = t_1\alpha_1 + \cdots + t_{i+1}\alpha_{i+1},$$

where $\alpha_1, \ldots, \alpha_{i+1} \in \mathbf{Z}A$. From the equality

$$t\gamma = \tau_1\beta_1 + \cdots + \tau_i\beta_i$$

we obtain the system

$$\begin{aligned}
\alpha_1\gamma &= \beta_1(1 - a_1) \\
\alpha_2\gamma &= \beta_1(1 - a_2) + \beta_2(1 - a_1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\alpha_i\gamma &= \beta_{i-1}(1 - a_2) + \beta_i(1 - a_1) \\
\alpha_{i+1}\gamma &= \beta_i(1 - a_2).
\end{aligned}$$

If γ and $1-a_1$ are mutually coprime then $\beta_1 = \beta_1'\gamma, \ldots, \beta_i = \beta_i'\gamma$. Hence, $t = \tau_1\beta_1' + \cdots + \tau_i\beta_i'$. If $\gamma = 1 - a_1$ then we have

$$\begin{aligned}
\alpha_1 &= \beta_1'(1 - a_1) \\
\alpha_2 &= \beta_1'(1 - a_2) + \beta_2'(1 - a_1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\alpha_i &= \beta_{i-1}'(1 - a_2) + \beta_i'(1 - a_1) \\
\alpha_{i+1} &= \beta_i'(1 - a_2).
\end{aligned}$$

Consequently, $t = \tau_1\beta_1' + \cdots + \tau_i\beta_i'$. The lemma is proven.

For all nonnegative m and n, define the function

$$F(m,n) = \begin{cases} n & \text{if } m = 0,\ n \geq 0, \\ n+2 & \text{if } m = 1,\ n \geq 1, \\ 2n+2 & \text{if } m = 2,\ n \geq 2, \\ F(m-2,n)+n & \text{if } m \geq 4 \text{ is even, } n \geq 2, \\ F(m-1,n)+1 & \text{if } m \geq 3 \text{ is odd, } n \geq 2, \\ m+2 & \text{if } m \geq 2,\ n = 1. \end{cases}$$

This formula implies that $F(m,n) = nm/2 + n + 2$ for even $m \geq 4$ and all $n \geq 2$ and $F(m,n) = n(m+1)/2 + 3$ for odd $m \geq 3$ and all $n \geq 2$.

Theorem 1. *Suppose that A_m is a free abelian group of rank m, T_n is a free $\mathbf{Z}A_m$-module of rank n, and $W_{m,n} = M(T_n, A_m)$. Then the topological dimension of $W_{m,n}$ is equal to $F(m,n)$.*

Proof. We first show that the group $W_{m,n}$ admits a u-sequence of length $F(m,n)$ and thus

$$\operatorname{tdim}(W_{m,n}) \geq F(m,n).$$

1. Suppose that $m = 0$, i.e., $W_{m,n} = A_n$ is a free abelian group of rank n. Then

$$A_n \to A_{n-1} \to \cdots \to A_1 \to 1$$

is a desired u-sequence of length $F(0,n)$. Clearly, $\operatorname{tdim}(W_{0,n}) = n$.

2. Suppose that $m = 1$, $n \geq 1$. Then

$$W_{1,n} \to W_{1,n-1} \to \cdots \to W_{1,1} \to A_2 \to A_1 \to 1$$

is a desired u-sequence of length $n + 2 = F(1,n)$. Prove that, in this case, the topological dimension of $W_{1,n}$ is equal to $n+2$.

Let $H = M(T,A)$ be a nonabelian u-group and $\alpha(H) = m$, $\beta(H) = n$. Easily, the abelianization H_{ab} of H is isomorphic to $T/T(1-A) \times A$. Since $\beta(H) = n$, the abelian group $T/T(1-A)$ contains at most n independent elements. Therefore, if A_r is a free abelian group which is an epimorphic image of H then its rank r is at most $n+m$.

Hence, every u-sequence of maximal length begins with the sequence

$$W_{1,n} \to W_{1,n-1} \to \cdots \to W_{1,p},$$

$1 \leq p \leq n$, and then the characteristic α decreases to zero, i.e., the sequence has the continuation

$$W_{1,p} \to A_{1+p} \to \cdots \to A_1 \to 1.$$

The length of an entire u-sequence for $W_{1,n}$ is equal to $n+2$. Since the length does not depend on p, we have $\operatorname{tdim}(W_{1,n}) = F(1,n)$ for all $n \geq 1$.

3. Assume that $m = 2$, $n \geq 2$. Denote by $t_1, \dots, t_n$ a basis of T_n and denote by a_1, a_2 a basis of A_2. Consider the elements

$$\tau_i = t_i(1 - a_1) + t_{i+1}(1 - a_2)$$

for $i = 1, \dots, n-1$. Put

$$\overline{T}_1 = \tau_1 \mathbf{Z}A_2,\ \overline{T}_2 = \tau_1 \mathbf{Z}A_2 + \tau_2 \mathbf{Z}A_2, \dots, \overline{T}_{n-1} = \sum_{j=1}^{n-1} \tau_j \mathbf{Z}A_2.$$

By Lemma 3, $\overline{T}_1, \dots, \overline{T}_{n-1}$ are isolated submodules in T. Therefore, $G_i = W_{2,n}/T_i$, $i = 1, \dots, n-1$, are u-groups. We arrived at the sequence

$$W_{2,n} \to G_1 \to \cdots \to G_{n-1}$$

which can be extended by the u-sequence

$$G_{n-1} \to A_{n+2} \to \cdots \to A_1 \to 1.$$

The length of the resulting u-sequence for $W_{2,n}$ is equal to $2n + 2 = F(2,n)$.

4. Suppose that $m \geq 4$ is even, $n \geq 2$. Consider the u-sequence

$$W_{m,n} \to G_{m,n-1} \to \cdots \to G_{m,1} \to W_{m-2,n} \to \dots,$$

where $\alpha(G_{m,i}) = m$, $\beta(G_{m,i}) = i$, and the groups $G_{m,i}$ are constructed by analogy with the case of $m = 2$ on using Lemma 3. The length of this sequence is equal to $n + F(m-2,n) = F(m,n)$.

5. Suppose that $m \geq 3$ is odd, $n \geq 2$. Consider the u-sequence

$$W_{m,n} \to W_{m-1,n} \to \dots.$$

Its length is equal to $1 + F(m-1, n) = F(m, n)$.

6. Assume that $m \geq 2$, $n = 1$. The sequence

$$W_{m,1} \to W_{m-1,1} \to \dots \to W_{1,1} \to A_2 \to A_1 \to 1$$

has length $m + 2$ and is of maximal length.

To finish the proof of the theorem, it suffices to validate the following bounds for the lengths of u-sequence.

Suppose that the u-group $G_{m,j}$ has the characteristics $\alpha(G_{m,j}) = m$, $\beta(G_{m,j}) = j$ and is a homomorphic image of $W_{m,n}$, $1 \leq j \leq n$, $2 \leq n$. Then every u-sequence of $G_{m,j}$ meets the following bounds on its length:

(1) $l(G_{0,j}) = j$;
(2) $l(G_{1,j}) \leq j + 2$;
(3) $l(G_{m,j}) \leq j + mn/2 + 2$ if $m \geq 2$ is even;
(4) $l(G_{m,j}) \leq j + (m-1)n/2 + 3$ if $m \geq 3$ is odd.

The rest of the proof of the theorem establishes (1)–(4).

Case 1: $m = 0$ is trivial.

Case 2: $m = 1$. Since the topological dimension of $W_{1,j}$ is equal to $j + 2$, bound 2 is obvious.

Case 3: $m = 2$, $n \geq 2$. The u-sequence constructed for $G_{2,j}$ has the beginning

$$G_{2,j} \to G_{2,j-1} \to \dots \to G_{2,p},$$

where $1 \leq p \leq j$. Then the possible continuations are $G_{2,p} \to G_{1,q}$ or $G_{2,p} \to G_{0,q}$. Consider each of the possibilities separately.

(a) $G_{2,p} \to G_{1,q}$. Since, by Proposition 9, $p \geq q$, we have

$$l(G_{2,j}) \leq j - p + 1 + q + 2 \leq j + 3 \leq j + n + 2$$

if $n \geq 2$.

(b) $G_{2,p} \to G_{0,q}$. Then $l(G_{2,j}) \leq j + n + 2$. Case 3 is analyzed completely.

Case 4: $m = 3$, $n \geq 2$. A u-sequence for $G_{3,j}$ that can have maximal length, has the beginning

$$G_{3,j} \to G_{3,j-1} \to \dots \to G_{3,p},$$

$1 \leq p \leq j$, and one of the following continuations.

(a) $G_{3,p} \to G_{2,q}$. We have

$$l(G_{3,j}) = j - p + 1 + l(G_{2,q}) \leq j - p + 1 + q + n + 2 \leq j + n + 3.$$

(b) $G_{3,p} \to G_{1,q}$. In this case,

$$l(G_{3,j}) = j - p + 1 + l(G_{1,q}) = j - p + 1 + q + 2 \leq j - 1 + n + 3 < j + n + 2.$$

(c) $G_{3,p} \to G_{0,q}$. In this case,

$$l(G_{3,1}) = j - p + 1 + q \leq j + q \leq j + n + 3.$$

Case 5: $m \geq 4$ is even. Assume that the inequalities hold for lesser m. Every u-sequence for $G_{m,j}$ that can have maximal length has the beginning

$$G_{m,j} \to G_{m,j-1} \to \dots \to G_{m,p}$$

and then one of the following continuations.

(a) $G_{m,p} \to G_{m-1,q}$. Then $l(G_{m,j}) = j - p + 1 + l(G_{m-1,q})$. Since $m-1$ is an odd less than m, we have $l(G_{m-1,q}) \leq q + (m-2)n/2 + 3$. Therefore,

$$l(G_{m,j}) \leq j - p + 1 + q + 3 \leq j + 2 + mn/2$$

for $n \geq 2$.

(b) $G_{m,p} \to G_{m-2,q}$. Since $m-2$ is odd and $n-2 \geq 2$, it follows that

$$l(G_{m,j}) \leq j - p + 1 + l(G_{m-2,q}) \leq j - 1 + 1 + q + (m-2)n/2 + 2 \leq j + 2 + mn/2.$$

(c) $G_{m,p} \to G_{m-r,q}$, $r \geq 3$, $1 \leq q \leq n$. We have

$$l(G_{m,j}) = j - p + 1 + l(G_{m-r,q}) \leq j - p + 1 + n + (m-r)n/2 + 3 \leq mn/2 + j + 2.$$

Case 6: $m \geq 5$ is odd.

As in Case 5, a u-sequence that is a candidate to be maximal has the beginning

$$G_{m,j} \to G_{m,j-1} \to \cdots \to G_{m,p},$$

$1 \leq p \leq j$, and then one of the following continuations.

(a) $G_{m,p} \to G_{m-1,q}$. By induction we then infer

$$l(G_{m,j}) = j - p + 1 + l(G_{m-1,q}) \leq j - p + 1 + q + (m-1)n/2 + 2 \leq j + 3 + (m-1)n/2,$$

since $q \leq p$.

(b) $G_{m,p} \to G_{m-2,q}$. In this case,

$$\begin{aligned} l(G_{m,j}) &= j - p + 1 + l(G_{m-2,q}) \leq j - p + 1 + l(W_{m-2,n}) \\ &\leq j - p + 1 + n + (m-3)n/2 + 3 \leq j + (m-1)n/2 + 3. \end{aligned}$$

(c) $G_{m,p} \to G_{m-r,q}$, $r \geq 3$. In this case, we have

$$l(G_{m,j}) = j - p + 1 + l(G_{m-r,q}) \leq j - p + 1 + n + (m-3)n/2 + 3 \leq j + (m-1)n/2 + 3.$$

The theorem is proven.

The following theorem makes it possible to calculate the nonabelian topological dimension for $W_{m,n}$.

For all nonnegative integers m and n, define the function

$$F_0(m,n) = \begin{cases} mn/2 + 1 & \text{for } n \geq 2 \text{ and even } m, \\ (m+1)n/2 & \text{for } n \geq 1 \text{ and odd } m, \\ m & \text{for } n = 1 \text{ and every } m. \end{cases}$$

Theorem 1′. *Let A_m be a free abelian group of rank m, let T_n be a free $\mathbf{Z}A_m$-module of rank n, and let $W_{m,n} = M(T_n, A_m)$. Then $\operatorname{tdim}_0(W_{m,n})$ is equal to $F_0(m,n)$.*

Proof is similar to the proof of Theorem 1 and is based on the following inequalities. Suppose that a u-group $G_{m,j}$ has the characteristics $\alpha(G_{m,j}) = m$, $\beta(G_{m,j}) = j$ $W_{m,n}$. Therefore,

(1) if $m = 2l$ then $\operatorname{tdim}_0(G_{m,j}) \leq (l-1)n + j + 1$;

(2) if $m = 2l+1$ then $\operatorname{tdim}_0(G_{m,j}) =\leq ln + j$.

The proof is finished by noting that the nonabelian u-sequences

$$W_{1,n} \to W_{1,n-1} \to \cdots \to W_{1,1} \to 1,$$

$$W_{2,n} \to W_{2,n-1} \to \cdots \to W_{2,1} \to W_{1,1} \to 1,$$

$$W_{2l,n} \to W_{2l-1,n} \to \ldots,$$

$$W_{2l+1,n} \to W_{2l+1,n-1} \to \cdots \to W_{2l+1,1} \to W_{2l-1,n} \to \ldots$$

have length $F_0(m,n)$.

Theorem 2. *The equality* $\mathrm{tdim}_0(G) = \mathrm{tdim}_0(G_{\mathrm{split}})$ *holds for every finitely generated nonabelian u-group G.*

Proof. We first prove that $\mathrm{tdim}_0(G) \geq \mathrm{tdim}_0(G_{\mathrm{split}})$. Let

$$G = G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} G_{n-1} \xrightarrow{\varphi_n} 1$$

be a sequence of nonabelian u-groups and epimorphisms. By Proposition 4, there exist homomorphisms $\psi_i : G_{i-1,\mathrm{split}} \to G_{i,\mathrm{split}}$ and an embedding $\varphi_i : G_{i-1} \to G_i$ such that the diagram

$$\begin{tikzcd}
G_0 \arrow[r, "\varphi_1"'] \arrow[d, "\alpha_0"] & G_1 \arrow[r, "\varphi_2"'] \arrow[d, "\alpha_1"] & \dots \arrow[r, "\varphi_{n-1}"'] \arrow[d] & G_{n-1} \arrow[r, "\varphi_n"'] \arrow[d, "\alpha_{n-1}"] & 1 \arrow[d, "\alpha_n"] \\
G_{0,\mathrm{split}} \arrow[r, "\psi_1"'] & G_{1,\mathrm{split}} \arrow[r, "\psi_2"'] & \dots \arrow[r, "\psi_{n-1}"'] & G_{n-1,\mathrm{split}} \arrow[r, "\psi_n"'] & 1
\end{tikzcd}$$

commutes.

Put $H_0 = G_{0,\mathrm{split}}$ and denote by H_i the image of H_{i-1} under ψ_i. Denote the restriction of ψ_i to H_{i-1} by $\widehat{\psi}_i$. We obtain the sequence of u-groups and epimorphisms

$$H_0 \xrightarrow{\widehat{\psi}_1} H_1 \xrightarrow{\widehat{\psi}_2} \dots \xrightarrow{\widehat{\psi}_{n-1}} H_{n-1} \xrightarrow{\widehat{\psi}_n} 1.$$

Show that $H_1, \dots, H_{n-1}$ are nonabelian u-groups and the kernels of the epimorphisms $\widehat{\psi}_i$ are nontrivial.

Since $G_i^{\alpha_i} \leq H_i \leq G_{i,\mathrm{split}}$; therefore, H_i are nonabelian u-groups. Let g_{i-1} be a nonidentity element in $\ker \varphi_i$. Then $g_{i-1}^{\alpha_{i-1}}$ is a nonidentity element of H_i that lies in the kernel of $\widehat{\psi}_i$.

Show that

$$\mathrm{tdim}_0(G_{\mathrm{split}}) \geq \mathrm{tdim}_0(G).$$

Put $T_0 = \mathrm{Fit}(G)$, $A_0 = G/T_0$. Consider the nonabelian u-sequence

$$M(T_0, A_0) \xrightarrow{\varphi_1} M(T_1, A_1) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} M(T_n, A_n). \tag{1}$$

Since $M(T_n, A_n)$ is a nonabelian group, it contains an element h_n not belonging to its Fitting radical. Let g_0' be a preimage of h_n in $M(T_0, A_0)$. The element g_0' has the form

$$\begin{pmatrix} a_0 & 0 \\ t_0 & 1 \end{pmatrix},$$

where $1 \neq a_0 \in A_0$, $t_0 \in T_0$. Choose an element g_0 in G so that its image in A_0 be equal a_0.

Consider the splittable envelope and standard embedding of G corresponding to g_0, i.e.,

$$\alpha : g \mapsto \begin{pmatrix} \bar{g} & 0 \\ [g, g_0] & 1 \end{pmatrix}.$$

The epimorphisms φ_i induce epimorphisms of abelian groups $\overline{\varphi}_i : A_{i-1} \to A_i$ and module epimorphisms $\widetilde{\varphi}_i : T_{i-1} \to T_i$ that agree with the corresponding $\overline{\varphi}_i$'s.

Consider the epimorphisms $\psi_1, \dots, \psi_n$ defined as follows:

$$\psi_1 : \begin{pmatrix} \bar{g} & 0 \\ [g, g_0] & 1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{g}^{\overline{\varphi}_1} & 0 \\ [g, g_0]^{\widetilde{\varphi}_1} & 1 \end{pmatrix},$$

$$\psi_2 : \begin{pmatrix} \bar{g}^{\overline{\varphi}_1} & 0 \\ [g, g_0]^{\widetilde{\varphi}_1} & 1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{g}^{\overline{\varphi}_1\overline{\varphi}_2} & 0 \\ [g, g_0]^{\widetilde{\varphi}_1\widetilde{\varphi}_2} & 1 \end{pmatrix}, \quad \text{etc.}$$

Put $G_i = G^{\alpha\psi_1\dots\psi_i}$. We obtain the sequence

$$G = G_0 \xrightarrow{\psi_1} G_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{n-1}} G_{n-1} \xrightarrow{\psi_n} G_n.$$

Show that all groups in this sequence are nonabelian u-groups and all epimorphisms have nontrivial kernels.

Indeed, suppose that t_n is a nonzero element in the Fitting radical of $M(T_n, A_n)$ and t_0 is its preimage in $M(T_0, A_0)$. The element $t_0^{g_0-1}$ belongs to the Fitting radical of G and is nonzero. Its image is nonzero in the Fitting radical of $M(T_n, A_n)$ by the choice of g_0 and the fact that $M(T_n, A_n)$ is a u-group. Furthermore, the image of g_0 in $M(T_n, A_n)$ has the form $\begin{pmatrix} a_n & 0 \\ * & 1 \end{pmatrix}$, where $a_n \neq 1$. Therefore, $M(T_n, A_n)$ is a nonabelian group.

We are left with proving that the kernels of the ψ_i's are nontrivial. The epimorphisms φ_i either decrease the value of α or leave it unchanged but, in this case, they diminish the value of β. Since the parameters $\alpha(G_i)$ and $\alpha(H_i)$ coincide, in the first case, ψ_i has a nonidentity kernel. If α is not changed under ψ_i then the kernel $\ker\varphi_i$ has a nontrivial intersection with the Fitting radical of $M(T_{i-1}, A_{i-1})$. Suppose that $1 \neq t \in \ker\varphi_i$. Then $1 \neq t^{g_0-1} \in \ker\psi_i$. The theorem is proven.

Theorem 3. *Let G be a free metabelian group of rank $n \geq 2$. Then* $\mathrm{tdim}_0(G) = F_0(n, n-1)$.

Proof. Let $G' = T$ be the Fitting radical of G and put $G/T = A$. By Theorem 2,

$$\mathrm{tdim}_0(G) = \mathrm{tdim}_0(M(T, A)).$$

Consequently, the theorem is equivalent to the fact that

$$\mathrm{tdim}_0(M(T, A)) = \mathrm{tdim}_0(M(L, A)),$$

where L is a free $\mathbf{Z}A$-module of rank $n-1$.

Let

$$M(T, A) = M(T_0, A_0) \xrightarrow{\varphi_1} M(T_1, A_1) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_l} M(T_l, A_l) \tag{2}$$

be a sequence of nonabelian u-groups and epimorphisms.

Suppose that $x_1, \dots, x_n$ is a basis of G.

Since $M(T_l, A_l)$ is a nonabelian group, the image of a generator a_1 of A is mapped to a nonidentity element of A_l. Denote by a_1 the image of x_1 in $A = G/T$.

The Magnus embedding [5] implies that the system of elements $\{[x_1, x_2], \dots, [x_1, x_n]\}$ generates a free $\mathbf{Z}A$-module and is its base. Take this submodule T as L and the embedding $\alpha(x_1)$ as the embedding of G into its splittable envelope $M(T, A)$.

All elements x_i, x_j, x_m in G satisfy the relation

$$[x_i, x_j]^{1-x_m}[x_j, x_m]^{1-x_i}[x_m, x_i]^{1-x_j} = 1.$$

Therefore, the module T and its submodule L meet the inclusion $T(1-a_1) \leq L$.

Each of the φ_i's induces an epimorphism $\overline{\varphi} : A_{i-1} \to A_i$ of abelian groups and a module epimorphism $\widetilde{\varphi}_i : T_{i-1} \to T_i$ that agrees with $\overline{\varphi}$.

Put $L = L_0$, $L_i = L_{i-1}^{\widetilde{\varphi}_i}$. Denote by $\widehat{\varphi}_i$ the restriction of φ_i to the subgroup $M(L_{i-1}, A_{i-1})$. We obtain the sequence

$$M(L, A) = M(L_0, A_0) \xrightarrow{\widehat{\varphi}_1} M(L_1, A_1) \xrightarrow{\widehat{\varphi}_2} \dots \xrightarrow{\widehat{\varphi}_l} M(L_l, A_l)$$

of nontrivial groups, since $T_{i-1}(1-a_1) \leq L_{i-1}$ and $T_{i-1} \neq 0$.

Show that the kernels of all $\widehat{\varphi}_i$ are nontrivial. The epimorphisms φ_i are nontrivial. Therefore, $\overline{\varphi}_i$ or $\widetilde{\varphi}_i$ is a nontrivial epimorphism. Consider the two cases:

1. $\ker(\widetilde{\varphi}_i) \neq 0$. Suppose that $0 \neq t \in \ker(\widetilde{\varphi}_i)$. Since $T_{i-1}(1-a_1) \leq L_{i-1}$, we have

$$0 \neq t(1-a_1) \in \ker(\widetilde{\varphi}_i) \cap L_{i-1} \leq \ker(\hat{\varphi}_i).$$

2. $\ker(\bar{\varphi}_i) \neq 1$. In this case, the rank of A_i is less than that of A_{i-1}. Hence, $\ker(\hat{\varphi}_i) \neq 1$.
Thus, we have proven the inequality $\mathrm{tdim}_0(G) \leq F_0(n, n-1)$.
Prove the reverse inequality. Let

$$M(F, A) = M(F_0, A_0) \xrightarrow{\psi_1} M(F_1, A_1) \xrightarrow{\psi_2} \dots \xrightarrow{\psi_l} M(F_l, A_l)$$

be a nonabelian u-sequence, where F is a free $\mathbf{Z}A$-module of rank $n-1$ and A is a free abelian group of rank n.

Since $M(F_l, A_l)$ is a nonabelian group, the image of some generator a_1 of A differs from the nonidentity of A_l. The element a_1 is the image of the generator x_1 under the homomorphism $G \to G/T = A$.

Consider the standard embedding $\alpha(x_1) : G \to M(T, A)$. Let L be the submodule in T defined in the first part of the proof. Then $L(1-a_1) \leq T(1-a_1) \leq L$. Therefore, the $u_{\times}$-groups meet the inclusions

$$M(L(1-a_1), A) \leq M(T(1-a_1), A) \leq M(L, A);$$

moreover, the groups $M(L, A)$ and $M(L(1-a_1), A)$ are isomorphic. Let

$$M(L, A) \xrightarrow{\varphi_1} M(R, B) \xrightarrow{\varphi_2} \dots$$

be a nonabelian u-sequence. It induces the sequences

$$\begin{gathered} M(T(1-a_1), A) \xrightarrow{\varphi_1'} M((T(1-a_1))^{\widetilde{\varphi}_1}, B) \xrightarrow{\varphi_2'} \dots, \\ M(L(1-a_1), A) \xrightarrow{\varphi_1''} M(R(1-a_1)^{\overline{\varphi}_1}, B) \xrightarrow{\varphi_2''} \dots. \end{gathered} \tag{3}$$

The sequence (3) consists of nonabelian u-groups and nontrivial epimorphisms. The theorem is proven.

Using Theorems 1 and 1′ and their proofs, define the topological dimension of a free metabelian group.

Theorem 4. *Let G be a free metabelian group of rank $n \geq 2$. Then*

$$\mathrm{tdim}(G) = \begin{cases} F(n, n-1) - 1 & \text{if } n \geq 4 \text{ is even}, \\ F(n, n-1) & \text{if } n \geq 1 \text{ is odd}, \\ 4 & \text{if } n = 2. \end{cases}$$

Proof. Suppose that $n \geq 3$ and $\alpha(x_1) : G \to M(T_{n-1}, A_n)$, where $\bar{x}_1, \dots, \bar{x}_n$ is a basis of A_n, $t_i = [x_i, x_1]$, $i = 2, \dots, n$, is a basis of the free module T_{n-1}. Then

$$x_i^\alpha = \begin{pmatrix} \overline{x}_i & 0 \\ [x_i, x_1] & 1 \end{pmatrix}, \quad i = 2, \dots, n.$$

Suppose that $n \geq 4$ is even. Consider a nonabelian u-sequence for the group $W_{n,n-1} = M(T_{n-1}, A_n)$ that passes through all groups $W_{i,j}$ for even i, $2 \leq i \leq n$, and all $1 \leq j \leq n-1$. It is easy to verify that the length of this sequence is maximal, i.e., is equal to $F_0(n, n-1)$. Continue it with the abelianization of $W_{2,n-1}$. Thus, the abelian part of the sequence begins with the group $B_{n+1} = \langle \bar{x}_{n-1}, \bar{x}_n, t_1, \dots, t_{n-1} \rangle$. The image of the group G in B_{n+1} coincides with $B_n = \langle \bar{x}_n, t_1, \dots, t_{n-1} \rangle$. Since $F(n, n-1) - F_0(n, n-1) = n+1$, it follows that $\mathrm{tdim}(G) \geq F(n, n-1) - 1$. Granted the inequality $\mathrm{tdim}(G) \leq \mathrm{tdim}_0(G) + n$, we obtain the desired result.

In the case when $n \geq 3$ is odd, the proof follows from two remarks. First, the definition implies that $F(n, n-1) - F_0(n, n-1) = 2$. Second, it is necessary to prove that the topological dimension of the subgroup G cannot exceed the dimension of the group $M(T_{n-1}, A_n)$. This result implies that $\mathrm{tdim}(G) = F(n, n-1)$.

To check the inequality $\mathrm{tdim}(G) \leq \mathrm{tdim}(M(T_{n-1}, A_n))$, observe that, in proving Proposition 6, for every epimorphism $\varphi : G_1 \to G_2$ of a nonabelian u-group, we constructed a homomorphism $\varphi^* : G_{1,\mathrm{split}} \to G_{2,\mathrm{split}}$; moreover, $\ker \varphi^*$ is the splitting of the kernel of φ (in coordinates). Furthermore, recall that the characteristics of the groups G_i and $G_{i,\mathrm{split}}$ coincide.

This proposition extends obviously to the case when G_2 is an abelian group, provided that $\ker \varphi$ includes $\mathrm{Fit}(G_1)$. As above, denote the so-obtained homomorphism by φ^* and call it the *splitting* of φ.

Given a free metabelian group G, consider a u-sequence of maximal length $\mathrm{tdim}(G)$

$$G = G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_l} G_l \xrightarrow{\varphi_{l+1}} A_r \to \dots \to 1, \tag{4}$$

where A_r is a free abelian group of rank r and all preceding groups are nonabelian. From this sequence, construct the induced chain

$$G = G_{0,\mathrm{split}} \xrightarrow{\varphi_1^*} G_{1,\mathrm{split}} \xrightarrow{\varphi_2^*} \dots \xrightarrow{\varphi_l^*} G_{l,\mathrm{split}} \xrightarrow{\varphi_{l+1}^*} A_t \to \dots \to 1. \tag{5}$$

We need to prove that the rank of A_r is equal to the rank t of the free abelian group A_t. Put $\gamma = \varphi_1 \varphi_2 \dots \varphi_l \varphi_{l+1}$, i.e., $\gamma : G \to A_r$. Clearly, $\ker \gamma$ includes $G' = \mathrm{Fit}(G)$. Therefore, there exists $\gamma^* : G_{0,\mathrm{split}} \to A_r$. Clearly, $\gamma^* = \varphi_1^* \varphi_2^* \dots \varphi_l^* \varphi_{l+1}^*$, and $r = t$.

If $n = 2$ then a maximal chain of inequalities for the characteristics of u-groups is as follows:

$$(2,1) > (1,1) > (0,2) > (0,1) > (0,0).$$

It has length 4. Since a free metabelian group of rank 2 can be mapped homomorphically onto $W_{1,1}$, this implies that $\mathrm{tdim}(G) = 4$. The theorem is proven.

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