Algebra and Logic, Vol. 43, No. 3, 2004

METABELIAN PRODUCTS OF GROUPS

V. N. Remeslennikov and N. S. Romanovskii*

UDC 512.5

Key words: *metabelian group, metabelian product, u-group, derived subgroup, Fitting radical, strong semidomain.*

We prove a number of facts on metabelian products of metabelian groups, useful in algebraic geometry over groups. Namely, for a metabelian product of arbitrary metabelian groups, we look at the structure of a derived subgroup, and the Fitting radical; find criteria determining when a metabelian product of u-groups is again a u-group; and specify conditions under which a metabelian product of metabelian groups is a strong semidomain.

INTRODUCTION

This work has been accomplished within the framework of the project dealing in creating algebraic geometry for metabelian groups. The fundamentals of algebraic geometry over groups are presented in [1, 2]. There, in particular, for every fixed group G, the category of G-groups is introduced, the concepts of a free G-group in a group variety (quasivariety) and of zero divisors and domain are defined, and the necessity of these concepts in developing algebraic geometry over G is explained. In the language of those concepts, for instance, the criterion for algebraic sets being irreducible is formulated for the case where G is a torsion-free hyperbolic group.

Unfortunately, some of the notions, for example, that of a domain, are inapplicable to metabelian groups, for a non-trivial metabelian group always insists on non-trivial zero divisors (cf. Sec. 5). In the present paper, we make attempts to account for the specific character of metabelian groups, and to obtain a number of results with provision for furthering their applications to problems of creating algebraic geometry over metabelian groups.

First, for every two metabelian groups, we establish the structure of their metabelian product (Thms. 1-3); in particular, the theorems proved yield a description of the structure of free G-groups. In [3], note, the structure of coordinate groups of algebraic sets was described for a free metabelian group F of rank at least 2; specifically, there, the concept of a u-group was defined in terms of a group universally equivalent to F.

In Theorem 4, we formulate a criterion saying when the property of being a u-group is preserved under metabelian products, which can be appealed to to explicitly compute the coordinate group $G^n = G\times\ldots\times G$, $n \geqslant 1$.

In Sec. 5, concepts of a semidomain and of a strong semidomain for metabelian groups are defined as analogs of the notion of a domain in the general situation, and conditions are specified under which a metabelian product of two metabelian groups is a strong semidomain.

*Supported by RFBR grant No. 02-01-00293, and FP "Universities of Russia" grant No. 04.01.053.

Translated from *Algebra i Logika*, Vol. 43, No. 3, pp. 341-352, May-June, 2004. Original article submitted March 3, 2003.

1. AUXILIARIES

1.1. Let G be a group, with $a, b \in G$. Then $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. A derived subgroup of G is denoted by $[G, G]$. By $\mathrm{Fit}(G)$ we denote the Fitting radical of G, that is, a product of all normal nilpotent subgroups. $G_1 * G_2$ stands for the metabelian product of metabelian groups G_1 and G_2.

For a given group A, let the right ZA-module T be defined. Denote by $M(A, T)$ a group which is an extension of the additive group of T by A. It will be convenient to identify this group with a multiplicative group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$.

1.2. Let A be representable as a factor group F/R, where F is a free group with basis $\{x_i \mid i \in I\}$. Denote by a_i the canonical image of an element x_i in A. Consider a right free ZA-module T with basis $\{t_i \mid i \in I\}$ and a *Magnus homomorphism* $\varphi : F \to M(A, T)$ defined by the mapping $x_i \to \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix}$. We recall some of the known facts.

Proposition 1 [4]. The kernel of the homomorphism φ is $[R, R]$.

Therefore φ determines an embedding of the group $F/[R, R]$ into a group $M(A, T)$, which we refer to as the *Magnus embedding*.

Proposition 2 [4]. The matrix $\begin{pmatrix} a & 0 \\ \sum t_i u_i & 1 \end{pmatrix}$ lies in $F\varphi$ if and only if $a - 1 = \sum(a_i - 1)u_i$. In particular, $R\varphi$ is identified with an additive group of the submodule of T consisting of elements $\sum t_i u_i$ for which $\sum(a_i - 1)u_i = 0$.

Let $h = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ be an arbitrary element of $M(A, T)$. We distinguish its diagonal and unitriangular parts: $d(h) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $u(h) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

Proposition 3 [5]. Let H be an arbitrary subset of $F\varphi$ and $\bar{H}$ be the normal closure in $M(A, T)$ of all elements $d(h)$ and $u(h)$ whenever $h \in H$. Then $\bar{H} \cap F\varphi = H^{F\varphi}$.

2. THE STRUCTURE OF A DERIVED SUBGROUP FOR A METABELIAN PRODUCT OF METABELIAN GROUPS

Letting G_1 and G_2 be metabelian groups, we consider their metabelian product $G_1 * G_2$. Represent every group G_j $(j = 1, 2)$ as a factor group F_j/R_j, where F_j is a free metabelian group with basis $X_j = \{x_i \mid i \in I_j\}$. Then G is a free factor group F/R, where F is a free metabelian group with basis $X = X_1 \cup X_2 = \{x_i \mid i \in I = I_1 \cup I_2\}$ and R is the normal closure in F of the set $R_1 \cup R_2$. Let $\bar{F}_j = F_j/[F_j, F_j]$ and $\bar{F} = F/[F, F]$, assume that $\bar{x}_i$ $(i \in I)$ is the canonical image of an element x_i in $\bar{F}$, and suppose that T_j is a free $Z\bar{F}_j$-module with basis $\{t_i \mid i \in I_j\}$ and T is a free $Z\bar{F}$-module with basis $\{t_i \mid i \in I\}$. Consider a Magnus embedding of F into $M(\bar{F}, T)$ defined by the formula $x_i = \begin{pmatrix} \bar{x}_i & 0 \\ t_i & 1 \end{pmatrix}$ such that F_j embeds into $M(\bar{F}_j, T_j)$.

Proposition 3 implies that G embeds into the factor group of $M(\bar{F}, T)$ under the normal closure of elements $d(r)$ and $u(r)$, where $r \in R_1 \cup R_2$. We factor $M(\bar{F}, T)$ with respect to the normal closure of the diagonal elements $d(r)$. In our constructions, then, the free modules T_1, T_2, and T over the integral group rings of the groups $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}$, respectively, will be replaced by free modules (for which we reserve the same designations T_1, T_2, and T) over group rings of $A = G_1/[G_1, G_1]$, $B = G_2/[G_2, G_2]$, and $C = A \times B = G/[G, G]$. Notice that T (as a Z-module) decomposes into the direct sum

$$\sum_{b\in B}^{\oplus} T_1 \cdot b \oplus \sum_{a\in A}^{\oplus} T_2 \cdot a.$$

Let a_i ($i \in I_1$) be the canonical image of $x_i \in X_1$ in A and b_i ($i \in I_2$) be the canonical image of $x_i \in X_2$ in B. Put

$$L_1 = \left\{ \sum_{i\in I_1} t_i u_i \in T_1 \;\middle|\; \sum_{i\in I_1} (a_i - 1)u_i = 0 \right\},$$

$$L_2 = \left\{ \sum_{i\in I_2} t_i u_i \in T_2 \;\middle|\; \sum_{i\in I_2} (a_i - 1)u_i = 0 \right\}.$$

The images of elements of R_j ($j = 1, 2$) in $M(C, T)$ are represented by unitriangular matrices, and can be identified with a submodule $\bar{R}_j$ of T_j; moreover, $\bar{R}_j \leqslant L_j$ by Prop. 2. The normal closure of a set $\bar{R}_1 \cup \bar{R}_2$ in $M(C, T)$ is identified with the submodule $\bar{R}$ of a module T, which, treated as a Z-module, decomposes into the direct sum

$$\sum_{b\in B}^{\oplus} \bar{R}_1 \cdot b \oplus \sum_{a\in A}^{\oplus} \bar{R}_2 \cdot a.$$

Consider factor modules $P = T_1/\bar{R}_1$, $S = T_2/\bar{R}_2$, and $Q = T/\bar{R}$. Obviously, Q, treated as a Z-module, decomposes into the direct sum

$$\sum_{b\in B}^{\oplus} P \cdot b \oplus \sum_{a\in A}^{\oplus} S \cdot a.$$

In view of the above considerations, the group G is canonically embedded into $M(C, Q)$, in which case G_1 embeds into $M(A, P)$ and G_2 embeds into $M(B, S)$. We have

$$G_1/[G_1, G_1] \cong A, \quad G_2/[G_2, G_2] \cong B, \quad G/[G, G] \cong C.$$

Therefore

$$[G_1, G_1] = G_1 \cap \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} = G_1 \cap \begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix}$$

is identified with the submodule $P_0 = L_1/\bar{R}_1$ of a module P, $[G_2, G_2]$ — with the submodule $S_0 = L_2/\bar{R}_2$ of S, and $[G, G]$ — with some submodule Q_0 of Q. Notice that Q_0 contains

$$L_0 = \sum_{b\in B}^{\oplus} P_0 \cdot b \oplus \sum_{a\in A}^{\oplus} S_0 \cdot a$$

as a submodule. Clearly, the factor Q_0/L_0 is isomorphic to the derived subgroup of a product of Abelian groups A and B, treated as a ZC-module. Thus we have in fact proved the following:

THEOREM 1. Let $G = G_1 * G_2$ be a metabelian product of the metabelian groups

$$A = G_1/[G_1, G_1], \quad B = G_2/[G_2, G_2], \quad C = A \times B = G/[G, G].$$

Then the derived subgroup $[G, G]$ of G, treated as a ZC-module, contains a submodule H, which decomposes into a direct sum of Z-modules thus:

$$\sum_{b\in B}^{\oplus} [G_1, G_1] \cdot b \oplus \sum_{a\in A}^{\oplus} [G_2, G_2] \cdot a,$$

and the factor module $[G,G]/H$ is isomorphic to a derived subgroup of the metabelian product $A * B$.

THEOREM 2. Let A and B be Abelian groups, $C = A \times B$, and $G = A * B$. Then the derived subgroup $[G,G]$ of G is isomorphic as a ZC-module to $(A-1)(B-1) \cdot ZC$.

Proof. Let T be the right free module with basis $\{t_1, t_2\}$. Consider a Shmel'kin embedding (cf. [6, 7]) of G in $M(C,T)$, given by the map

$$a \to \begin{pmatrix} a & 0 \\ t_1(a-1) & 1 \end{pmatrix}, \quad b \to \begin{pmatrix} b & 0 \\ t_2(b-1) & 1 \end{pmatrix}, \quad a \in A,\ b \in B.$$

Then the derived subgroup $[G,G]$ of G is equal to $G \cap \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$ and is identified with a submodule L of T consisting of elements $t_1u_1 + t_2u_2$ and satisfying $u_1 \in (A-1) \cdot ZC$, $u_2 \in (B-1) \cdot ZC$, $u_1 + u_2 = 0$. Obviously, the projection of L into $t_1 \cdot ZC$ is an embedding. It is also easy to see that the image of L, under this embedding, is equal to $t_1(A-1)(B-1) \cdot ZC$. The theorem is proved.

3. FITTING RADICAL FOR A METABELIAN PRODUCT OF METABELIAN GROUPS

It is obvious that if G is a metabelian group then (in the additive notation)

$$\mathrm{Fit}(G) = [G,G] \cup \{g \in G \setminus [G,G] \mid (\exists n = n(g) \in N)\ [G,G](g-1)^n = 0\}.$$

Also, put

$$\mathrm{Fit}_\omega(G) = [G,G] \cup \{g \in G \setminus [G,G] \mid (\forall x \in [G,G])\,(\exists n = n(g,x) \in N)\ x(g-1)^n = 0\}.$$

The definition above implies that $\mathrm{Fit}_\omega(G)$ contains $\mathrm{Fit}(G)$ and is a maximal locally nilpotent subgroup of G containing the derived subgroup.

We say that a metabelian group G is *special* if G is a periodic 2-group, and the factor group $G/[G,G] = \langle a \rangle$ is a cyclic group of order 2. For such a group, $[G,G] = [G,G](a-1)$ and $[G,G] = 2[G,G]$ (since $(a-1)^2 = -2(a-1)$). Therefore the derived subgroup of a special group is a complete 2-group, and decomposes into a direct sum of quasicyclic 2-groups. If $[G,G] \neq 0$ then $[G,G](a-1)^{n+1} = [G,G](2^n(a-1)) = [G,G] \neq 0$, in which case G is not a nilpotent group. Clearly, a special group of bounded period has order 2. A non-trivial example of a special group is the extension of a quasicyclic 2-group by an automorphism mapping each element into its inverse.

THEOREM 3. Let $G = G_1 * G_2$ be the metabelian product of non-trivial metabelian groups G_1 and G_2. Then:

(1) $\mathrm{Fit}_\omega(G)$ is strictly larger than $[G,G]$ iff both of the factors are special groups;

(2) $\mathrm{Fit}(G)$ is strictly larger than $[G,G]$ iff both of the factors are cyclic groups of order 2.

Proof. Put $A = G_1/[G_1,G_1]$, $B = G_2/[G_2,G_2]$, and $C = A \times B = G/[G,G]$. We assume, for instance, that G_1 is not special, and prove that $\mathrm{Fit}_\omega(G) = [G,G]$. Consider an element $g \in G \setminus [G,G]$, letting its projection onto C be equal to $c = ab$, where $a \in A$ and $b \in B$.

First we handle the case where one of the following conditions holds: (1) one of the elements a or b is equal to 1; (2) $a \neq 1$, $b \neq 1$, and one of the groups A or B is not cyclic of order 2. Consider a group $A * B$. By Theorem 2, its derived subgroup (as a ZC-module) is isomorpic to $(A-1)(B-1) \cdot ZC$.

Suppose that condition (1) is satisfied, letting $a = 1$ and $c = b \neq 1$ for instance. If $1 \neq a' \in A$ then $(a' - 1)(b - 1)(c - 1)^n = (a' - 1)(b - 1)^{n+1} \neq 0$, for any natural n. Therefore $g \notin \mathrm{Fit}_\omega(G)$.

Assume that condition (2) holds, letting $|A| > 2$ for instance. If $|c| = \infty$ then $(a-1)(b-1)(c-1)^n \neq 0$, for every natural n; so, $g \notin \mathrm{Fit}_\omega(G)$. Suppose that the order of c is finite. Without loss of generality, we can assume that it is equal to a prime p. Then $|a| = |b| = p$. If $p = 2$ then we choose an element $a' \in A$ distinct from a and 1. We have $\langle a', b, c\rangle = \langle a'\rangle \times \langle b\rangle \times \langle c\rangle$, whence $(a'-1)(b-1)(c-1)^n \neq 0$, for any natural n. Let $p > 2$; then $\langle b, c\rangle = \langle b\rangle \times \langle c\rangle$ and $(a-1)(b-1)(c-1)^n = (cb^{-1}-1)(b-1)(c-1)^n = (c+1-b-cb^{-1})(c-1)^n \neq 0$, since $1, b, b^{-1}$ freely generate a free $Z\langle c\rangle$-module. Again we conclude that $g \notin \mathrm{Fit}_\omega(G)$.

We handle the last case where $a \neq 1$, $b \neq 1$, $|A| = 2$, and $|B| = 2$. By the hypothesis above, the group G_1 is not special, and $[G_1, G_1]$ contains an element x whose order is other than a degree of the number 2. Then $x(c - 1)^n = x(-2)^{n-1}(c - 1) = x(-2)^{n-1}c - x(-2)^{n-1} \neq 0$ for every natural n, since $x(-2)^{n-1}c = x(-2)^{n-1}ab \in [G_1, G_1]b$, $-x(-2)^{n-1} \in [G_1, G_1]$, and by Theorem 1, the sum of Z-modules $[G_1, G_1]b$ and $[G_1, G_1]$ is direct. Therefore $g \notin \mathrm{Fit}_\omega(G)$.

Below we assume that both G_1 and G_2 are special groups, and that G_1 is not cyclic of order 2. We claim that $\mathrm{Fit}(G) = [G, G]$. Let $A = \langle a\rangle$, $B = \langle b\rangle$, $g \in \mathrm{Fit}(G)$, and c be the canonical image of g in C. If the projection of c onto A is equal to a then $\langle g, [G, G]\rangle \equiv G_1 \bmod G_2^G$. The group G_1, and so also $\langle g, [G, G]\rangle$, will not be nilpotent, a contradiction. Suppose $c = b$. By Theorem 1, $[G_1, G_1](b - 1) \cong [G_1, G_1]$, whence $[G_1, G_1](b - 1)^{n+1} = [G_1, G_1](2^n(b - 1)) \neq 0$ for any $n \in N$, which is a contradiction with $\langle g, [G, G]\rangle$ being nilpotent. Thus $g \in [G, G]$.

Let G_1 and G_2 both be special, $A = \langle a\rangle$, and $B = \langle b\rangle$. We prove that $\mathrm{Fit}_\omega(G) > [G, G]$. Choose in G an element g so that its projection c onto C is equal to ab. Obviously, $g \notin [G, G]$. We claim that $g \in \mathrm{Fit}_\omega(G)$. The latter inclusion is equivalent to the fact that for any $x \in [G, G]$ ($[G, G]$ is treated as a ZC-module), there exists a natural n for which $x(c - 1)^n = 0$. Notice that $[G, G](c - 1) \leqslant H$ (in the notation of Theorem 1), which follows from the observation that by Theorem 2, the ZC-module $[G, G]/H$ is isomorphic to $(a-1)(b-1) \cdot ZC$, and $(a-1)(b-1)(c-1) = 0$. Therefore we may assume that $x \in H$. By Theorem 1, H is a 2-group. Let $|x| = 2^n$. We have $x(c-1)^{n+1} = x(-2)^n(c-1) = 0$. Thus $\mathrm{Fit}(G) > [G, G]$.

Following the argument above, we can show that if $G_1 = A = \langle a\rangle$ and $G_2 = B = \langle b\rangle$ are cyclic groups of order 2 then $ab \in \mathrm{Fit}(G) \setminus [G, G]$. The theorem is proved.

4. METABELIAN PRODUCT OF u-GROUPS

Recall that universal theories for free metabelian groups of ranks at least 2 coincide (cf. [3, 8]). A group that is universally equivalent to a free metabelian group of rank at least 2 is called a *u-group*. We have the following abstract characterization of u-groups.

Proposition 4. A metabelian group G is a u-group if and only if $\mathrm{Fit}(G)$ is an isolated Abelian subgroup distinct from G which, treated as a $Z[G/\mathrm{Fit}(G)]$-module, is torsion free.

[9] contains an example where a metabelian product of two u-groups is not a u-group. We argue for the following:

THEOREM 4. Let G_1 and G_2 be u-groups. Their metabelian product $G = G_1 * G_2$ is again a u-group if and only if $\mathrm{Fit}(G_1) = [G_1, G_1]$ and $\mathrm{Fit}(G_2) = [G_2, G_2]$.

Proof. Let $\mathrm{Fit}(G_1) > [G_1, G_1]$, $g \in \mathrm{Fit}(G_1) \setminus [G_1, G_1]$. Since $[G, G] \cap G_1 = [G_1, G_1]$, we have $g \notin [G, G]$. By Theorem 3, $\mathrm{Fit}(G) = [G, G]$. The element $g - 1$ acts trivially on $[G_1, G_1]$, and by Proposition 4, G cannot be a u-group.

Let $\mathrm{Fit}(G_1) = [G_1, G_1]$ and $\mathrm{Fit}(G_2) = [G_2, G_2]$. We prove that G is a u-group. In view of Proposition 4, the Abelian groups $A = G_1/[G_1, G_1]$ and $B = G_2/[G_2, G_2]$ are torsion free. Put $C = A \times B$. We need to show that $[G, G]$, treated as a ZC-module, is torsion free. Let $0 \neq x \in [G, G]$, $0 \neq \alpha \in ZC$. Here, we make use of the notation and statement of Theorem 1. If x does not belong to H then, in the factor module $[G, G]/H$, x is a non-trivial element. By Theorem 2, the factor module at hand is isomorphic to a module $(A - 1)(B - 1) \cdot ZC$. It remains to notice that the latter lacks in torsion since A and B are torsion free.

Suppose that $x \in H$, $x = u_1 + u_2$, where

$$u_1 \in \sum_{b\in B}^{\oplus} [G_1, G_1] \cdot b, \quad u_2 \in \sum_{a\in A}^{\oplus} [G_2, G_2] \cdot a.$$

To be specific, let $u_1 \neq 0$. We order B. Let

$$u_1 = v_1 b_1 + \ldots + v_m b_m, \quad \alpha = \alpha_1 b'_1 + \ldots + \alpha_n b'_n,$$

where

$$v_i \in [G_1, G_1], \quad v_m \neq 0, \quad \alpha_j \in ZA, \quad \alpha_n \neq 0, \quad b_i, b'_j \in B,$$
$$b_1 < \ldots < b_m, \quad b'_1 < \ldots < b'_n.$$

Since the module $[G_1, G_1]$ is ZA-torsion free, the leading term of an element $u_1\alpha$, which is equal to $v_m \alpha_n b_m b'_n$, is distinct from zero. Hence $x\alpha \neq 0$. The theorem is proved.

5. METABELIAN PRODUCT OF STRONG SEMIDOMAINS

Recall some of the definitions from [1]. Non-trivial elements x and y of G are called *zero divisors* if $[x^G, y^G] = 1$. A group without zero divisors is called a *domain*. Obviously, if a group has a non-trivial normal Abelian subgroup then such a group cannot be a domain. We can also assert that the non-trivial elements of $\mathrm{Fit}(G)$ are zero divisors in G. A metabelian group G is called a *semidomain* if the set of zero divisors coincides with $\mathrm{Fit}(G) \setminus \{1\}$. If, in addition, $\mathrm{Fit}(G) = [G, G]$, then the group is referred to as a *strong semidomain*.

Proposition 5. A metabelian non-Abelian group G is a strong semidomain if and only if $1 \neq x \in [G, G]$ and $y \in G \setminus [G, G]$ imply $[x, y] \neq 1$.

Proof. Let $[x, y] = 1$, for some non-trivial elements $x \in [G, G]$ and $y \in G \setminus [G, G]$. Then $[x, y^g] = [x, y[y, g]] = [x, y][x, [y, g]] = 1$, for any $g \in G$. Hence $[x, y^G] = 1$, and consequently y is a zero divisor in G; so, G is not a strong semidomain.

Now, let $1 \neq x \in [G, G]$ and $y \in G \setminus [G, G]$ imply $[x, y] \neq 1$. Suppose $1 \neq a, b \in G$ and $[a^G, b^G] = 1$. We need to prove that $a, b \in [G, G]$. If one of a, b lies in $[G, G]$, then, in view of the condition stated above, the other is also contained in $[G, G]$. Assume $a, b \notin [G, G]$. Let $1 \neq x \in [G, G]$; then $[a, x] \neq 1$ and $[a, x, b] \neq 1$, whence $[a^x, b] = [a[a, x], b] = [a, b][a, x, b] \neq 1$, a contradiction with $[a^G, b^G] = 1$. The proposition is proved.

Example. It is easy to see that a permutation group of degree 3, S_3, is a strong semidomain. Consider a metabelian product $G = G_1 * G_2$ of the groups G_1 and G_2 of which each is isomorphic to S_3. Fix second-order elements $a \in G_1$ and $b \in G_2$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Notice that S_3 factors into a semidirect product of an order 2 cyclic subgroup and its derived subgroup. Consequently, G factors into the semidirect product of $A * B$ and the normal closure of the derived subgroups of the groups G_1 and G_2. In particular, G contains $A * B$ as a subgroup. Based on Theorem 3, we can assert that $A * B$ has zero divisors not in the derived subgroup, namely, ab permutes with any element of the derived subgroup of the group $A * B$.

The element ab is a zero divisor in G, too, and it is not contained in $\mathrm{Fit}(G)$, since $\mathrm{Fit}(G) = [G, G]$ by Theorem 3. Thus the metabelian product of strong semidomains is not necessarily a semidomain.

THEOREM 5. (1) Let the metabelian product $G = G_1 * G_2$ of two metabelian groups G_1 and G_2 be a strong semidomain. Then every factor is either an Abelian group or a strong semidomain.

(2) Let each of the groups G_1 and G_2 be either a non-trivial torsion-free Abelian group or a strong semidomain such that the factor group with respect to its derived subgroup is torsion free. Then the metabelian product $G = G_1 * G_2$ is a strong semidomain.

Proof. (1) Let $[G_1, G_1] \neq 1$. Appealing to Proposition 5, we prove that G_1 is a strong semidomain. Let $1 \neq x \in [G_1, G_1]$ and $y \in G_1 \setminus [G_1, G_1]$. Since $x \in [G, G]$, $y \in G \setminus [G, G]$, and G is a strong semidomain, we have $[x, y] \neq 1$. Hence G_1 is a strong semidomain.

(2) Put $A = G_1/[G_1, G_1]$, $B = G_2/[G_2, G_2]$, and $C = A \times B$. Let $1 \neq x \in [G, G]$ and $y \in G \setminus [G, G]$. We need to show that $[x, y] \neq 1$. In the additive language, this means that $x(c-1) \neq 0$, where c is the canonical image of y in C. We use the notation of Theorem 1. If $x \notin H$ then the problem reduces to the case where $G_1 = A$ and $G_2 = B$, and by Theorem 2, the inequality $x(c-1) \neq 0$ holds since the group ring ZC has no zero divisors.

Let $x \in H$. As in the proof of Theorem 4, we decompose x into the sum $u_1 + u_2$, where

$$u_1 \in \sum_{b \in B}^{\oplus} [G_1, G_1] \cdot b, \quad u_2 \in \sum_{a \in A}^{\oplus} [G_2, G_2] \cdot a.$$

We can assume, for instance, that

$$u_1 = v_1 b_1 + \ldots + v_m b_m \neq 0, \ v_i \in [G_1, G_1], \ v_m \neq 0, \ b_i \in B, \ b_1 < \ldots < b_m.$$

If $c \in A$, then $v_m(c-1) \neq 0$, since G_1 is a strong semidomain, and so $x(c-1) \neq 0$. If $c = ab$, where $a \in A$ and $b \in B$, $b > 1$, then $0 \neq v_m a b_m b$ is a leading term in the decomposition of $u_1(c-1)$, yielding $x(c-1) \neq 0$. If $1 > b$ then the leading term in the decomposition of $u_1(c-1)$ is $-v_m b_m$, yielding $x(c-1) \neq 0$ again. The theorem is proved.

REFERENCES

1. G. Baumslag, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over groups," *J. Alg.*, **219**, No. 1, 16-79 (1999).
2. A. Myasnikov and V. N. Remeslennikov, "Algebraic geometry over groups. II. Logical foundations," *J. Alg.*, **234**, No. 1, 225-276 (2000).
3. V. Remeslennikov and R. Stohr, "On the quasivariety generated by a non-cyclic free metabelian group," preprint No. 2001/20, Manchester Centre Pure Math., Manchester (2001).
4. V. N. Remeslennikov and V. G. Sokolov, "Some properties of a Magnus embedding," *Algebra Logika*, **9**, No. 5, 566-578 (1970).
5. N. S. Romanovskii, "Algorithmic problems for solvable groups," *Algebra Logika*, **13**, No. 1, 26-34 (1974).
6. A. L. Shmel'kin, "Free products of groups," *Mat. Sb.*, **79**(**121**), No. 4(8), 616-620 (1969).
7. N. S. Romanovskii, "Shmel'kin embeddings for abstract and profinite groups," *Algebra Logika*, **38**, No. 5, 598-612 (1999).

8. O. Chapuis, "∀-free metabelian groups," *J. Symb. Log.*, **62**, No. 1, 159-174 (1997).
9. E. I. Timoshenko, "Universal theories of metabelian groups and the Shmel'kin embedding," *Sib. Mat. Zh.*, **42**, No. 5, 1168-1175 (2001).