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IRREDUCIBLE ALGEBRAIC SETS IN METABELIAN GROUPS

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We present the construction for a u-product $G_1 \circ G_2$ of two u-groups G_1 and G_2, and prove that $G_1 \circ G_2$ is also a u-group and that every u-group, which contains G_1 and G_2 as subgroups and is generated by these, is a homomorphic image of $G_1 \circ G_2$. It is stated that if G is a u-group then the coordinate group of an affine space G^n is equal to $G \circ F_n$, where F_n is a free metabelian group of rank n. Irreducible algebraic sets in G are treated for the case where G is a free metabelian group or wreath product of two free Abelian groups of finite ranks.

INTRODUCTION

In the present paper, we study into algebraic geometry over a non-commutative u-group, that is, a finitely generated metabelian group whose universal theory is the same as is one for a free metabelian group of rank at least two. The basics of algebraic geometry over groups are outlined in [1, 2]. If G is a group then the set of solutions for some system of equations in $x_1, \ldots, x_n$ with coefficients from G is called an *algebraic subset* of an affine space G^n. Algebraic sets, which are taken as a subbasis for a system of closed subsets, are used to define the Zariski topology on G^n. A group G is said to be *Noetherian over equations* if every system over it is equivalent to a finite system. On G^n, in this instance, the Zariski topology is Noetherian, that is, any closed set is uniquely decomposed into a finite non-cancellable union of irreducible algebraic sets.

Subject to the condition that G has no zero divisors, the union of two algebraic subsets in G^n is again an algebraic set. (Non-trivial elements x and y are called *zero divisors* if $[x^G, y^G] = 1$.) Note that a non-trivial finitely generated metabelian group is Noetherian over equations, but is not a domain, that is, it has zero divisors. The latter circumstance sort of complicates the study of algebraic geometry over metabelian groups.

The main results of the paper are as follows. In the first part, we show how to construct a u-product $G_1 \circ G_2$ of two u-groups G_1 and G_2, and prove that $G_1 \circ G_2$ is also a u-group and that every u-group, which contains G_1 and G_2 as subgroups and is generated by these, is a homomorphic image of $G_1 \circ G_2$. It is stated that if G is a u-group then the coordinate group of an affine space G^n is equal to $G \circ F_n$, where F_n is a free metabelian group of rank n. In the second part, we deal with irreducible algebraic sets in G for

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the case where G is a free metabelian group or wreath product of two free Abelian groups of finite ranks. It is proved that an irreducible set $S \subseteq G$ should satisfy one of the following conditions: (1) $S = \{g\}$ is a singleton; (2) $S = G$; (3) $S = g \cdot \mathrm{Fit}(G)$ is a coset w.r.t. a Fitting subgroup; (4) the set S is canonically embedded in $A = G/\mathrm{Fit}(G)$, in which case its image coincides with A, or with union of a finite set and finitely many cosets w.r.t. a same cyclic subgroup of A.

Note that the second section of this paper is tightly linked with [3]. Besides, there is some overlapping with [4], where Proposition 10 was proved as well, but in a less natural and simple form.

1. THE NOTATION AND PRELIMINARY FACTS

Since we are primarily concerned with metabelian groups, $G_1 * G_2$ will stand for a metabelian product of metabelian groups G_1 and G_2. For a metabelian group G, we put $G[x_1, \ldots, x_n] = G * F_n$, where F_n is a free metabelian group with basis $\{x_1, \ldots, x_n\}$. Elements of $G[x_1, \ldots, x_n]$ may be conceived of as left parts of all equations $w(x_1, \ldots, x_n) = 1$ over G whose solutions are being searched for in G^n. By [1], an algebraic set $V \subseteq G^n$ is completely characterized by its coordinate group $\Gamma(V) = G[x_1, \ldots, x_n]/\mathrm{Rad}(V)$, where $\mathrm{Rad}(V)$ is an annihilator of V in $G[x_1, \ldots, x_n]$.

Every group in which G is distinguished as a subgroup is called a *G-group*.

Proposition 1 [1, 2]. Let the group G be Noetherian over equations, and let H be a finitely generated G-group. Then the following are equivalent:

(1) H is the coordinate group of an irreducible algebraic set over G;

(2) G and H are G-universally equivalent;

(3) H is G-discriminated by G.

Recall that the *Fitting subgroup* $\mathrm{Fit}(G)$ in G is a maximal locally nilpotent normal subgroup. For the case where G is a finitely generated metabelian group, $\mathrm{Fit}(G)$ coincides with a maximal nilpotent normal subgroup. For an Abelian group G, as distinct from its traditional definition, we put $\mathrm{Fit}(G) = 1$.

Let A be an Abelian group of finite rank k, and let T be a right free ZA-module of finite rank n. Denote by $M(A, T)$ the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, which can be conceived of as an extension of the additive group of T by A, or as a wreath product W_{nk} of two free Abelian groups of ranks n and k.

A finitely generated metabelian group is called a *u-group* if all sentences of universal theory for a free metabelian group of rank at least two are true on it. A commutative u-group is free Abelian, and a non-commutative one has the same universal theory as has a free metabelian group of rank at least two [5].

Proposition 2 [5]. For a finitely generated non-commutative metabelian group G, the conditions below are equivalent:

(1) G is a u-group, i.e., its universal theory coincides with one for a free metabelian group of rank at least two;

(2) G is embedded in a suitable group $M(A, T)$;

(3) $\mathrm{Fit}(G)$ is an isolated Abelian subgroup in G, which, if treated as a ZA-module, where $A = G/\mathrm{Fit}(G)$, is module torsion free.

Let G be a non-commutative u-group, and let S be an irreducible algebraic set in G^n and $\Gamma = \Gamma(S)$ be its coordinate group. By Proposition 1, Γ is also a u-group. Following [4], with Γ we associate a pair of non-negative integers $\chi(\Gamma) = (\chi_1(\Gamma), \chi_2(\Gamma))$, where $\chi_1(\Gamma)$ is the difference of ranks of free Abelian groups $\Gamma/\mathrm{Fit}(\Gamma)$ and $G/\mathrm{Fit}(G)$, and $\chi_2(\Gamma)$ is the difference of ranks of modules $\mathrm{Fit}(\Gamma)$ and $\mathrm{Fit}(G)$, treated over integral group rings of $\Gamma/\mathrm{Fit}(\Gamma)$ and $G/\mathrm{Fit}(G)$, respectively. Note that $\chi_i(\Gamma) \leqslant n$, $i = 1, 2$.

Proposition 3 [4]. Let G be a non-commutative u-group, and let S_1 and S_2 be irreducible algebraic sets in G^n, where S_1 is strictly contained in S_2. Then $\chi(\Gamma(S_1)) < \chi(\Gamma(S_2))$.

2. CONSTRUCTING A u-PRODUCT

Let G_1 and G_2 be metabelian groups and $F = G_1 * G_2$ its metabelian product. Suppose an Abelian normal subgroup H_i containing $[G_i, G_i]$ is distinguished in each one of G_i $(i = 1, 2)$. Put

$$P(G_1, G_2; H_1, H_2) = F/[H_1H_2[F, F], H_1H_2[F, F]].$$

The structure of a metabelian product of two metabelian groups was studied in [3]. Similarly to [3, Thm. 1], we prove the following:

Proposition 4. Let

$$G = P(G_1, G_2; H_1, H_2), \quad A = G_1/H_1, \quad B = G_2/H_2, \quad H = H_1H_2[G, G], \quad C = A \times B = G/H.$$

Then an Abelian normal subgroup H of G, treated as a ZC-module, contains H_0 as a submodule which decomposes into a direct sum of Z-modules, that is,

$$\sum_{b\in B}^{\oplus} H_1 \cdot b \oplus \sum_{a\in A}^{\oplus} H_2 \cdot a,$$

and the factor module H/H_0 is isomorphic to a commutant of the metabelian product $A * B$.

Proposition 5 [3]. Let A and B be Abelian groups, $C = A \times B$, and $G = A * B$. Then the commutant of G, treated as a ZC-module, is isomorphic to the ideal $(A - 1)(B - 1) \cdot ZC$ of a ring ZC, in which case the isomorphism is determined by a map $[a, b] \to (1 - a)(1 - b)$, $a \in A$, $b \in B$.

Let G_1 and G_2 be u-groups. We call the group $P(G_1, G_2; H_1, H_2)$, where $H_i = \mathrm{Fit}(G_i)$ $(i = 1, 2)$, a *u-product* of the groups G_1 and G_2 and denote it by $G_1 \circ G_2$.

Proposition 6. Let G_1 and G_2 be u-groups; then their u-product $G = G_1 \circ G_2$ is also a u-group, with $\mathrm{Fit}(G) = \mathrm{Fit}(G_1) \cdot \mathrm{Fit}(G_2) \cdot [G, G]$, and every u-group, which contains G_1 and G_2 as subgroups and is generated by these, is a homomorphic image of G.

The **proof** of the first fact makes use of Propositions 4 and 5, and is similar to a proof for the corresponding fact in [3, Thm. 4]. The second fact is easily deducible from the definition of a u-product.

Proposition 6 and the definitions can be combined to yield the following:

Proposition 7. The operation of taking u-products on the class of u-groups is associative.

Note also that a u-product of free metabelian groups coincides with a metabelian product of these groups and, therefore, is a free metabelian group.

Proposition 8. Suppose G_1 and G_2 are u-groups, $\widetilde{G}_i$ is a subgroup of G_i $(i = 1, 2)$, and either $\widetilde{G}_i$ is non-Abelian or $\widetilde{G}_i \cap \mathrm{Fit}(G_i) = 1$. Then $\widetilde{G} = \widetilde{G}_1 \circ \widetilde{G}_2$ is canonically embedded in $G = G_1 \circ G_2$.

Proof. Put $H_i = \mathrm{Fit}(G_i)$ and $\widetilde{H}_i = \mathrm{Fit}(\widetilde{G}_i)$. By the hypotheses and Proposition 2, it follows that $\widetilde{H}_i = H_i \cap \widetilde{G}_i$. Therefore $\widetilde{A} = \widetilde{G}_1/\widetilde{H}_1$ and $\widetilde{B} = \widetilde{G}_2/\widetilde{H}_2$ are subgroups in $A = G_1/H_1$ and $B = G_2/H_2$, respectively. Put $C = A \times B$ and $\widetilde{C} = \widetilde{A} \times \widetilde{B}$. Let H_0 be as in Proposition 4 and $\widetilde{H}_0$ be a corresponding subgroup in $\widetilde{G}$. Proposition 4 implies that H_0 and $\widetilde{H}_0$, treated as Z-modules, have the following direct decompositions:

$$H_0 = \sum_{b\in B}^{\oplus} H_1 \cdot b \oplus \sum_{a\in A}^{\oplus} H_2 \cdot a, \quad \widetilde{H}_0 = \sum_{b\in \widetilde{B}}^{\oplus} \widetilde{H}_1 \cdot b \oplus \sum_{a\in \widetilde{A}}^{\oplus} \widetilde{H}_2 \cdot a.$$

Clearly, $\widetilde{H}_0$ embeds in H_0. Moreover, by Proposition 5, H/H_0 is isomorphic to the additive group of an ideal $(A-1)(B-1)\cdot ZC$ of a ring ZC, and $\widetilde{H}/\widetilde{H}_0$ is isomorphic to the additive group of an ideal $(\widetilde{A}-1)(\widetilde{B}-1)\cdot Z\widetilde{C}$ of a ring $Z\widetilde{C}$; hence, $\widetilde{H}/\widetilde{H}_0$ embeds in H/H_0. Lastly, $\widetilde{G}/\widetilde{H}=\widetilde{C}$ embeds in $G/H=C$. All of these facts imply that $\widetilde{G}$ is embeddable in G. The proposition is proved.

3. THE COORDINATE GROUP OF A SET G^n

Let G be a u-group and F_n be a free metabelian group with basis $\{x_1,\ldots,x_n\}$. Put $G(x_1,\ldots,x_n) = G\circ F_n$. The last-mentioned group is thought of as being a G-group in which G is embedded canonically. In this section, we work to find some representation for $G(x_1,\ldots,x_n)$, and then prove the following:

THEOREM 1. Let G be a non-Abelian u-group. Then $G(x_1,\ldots,x_n)$ is the coordinate group of a set G^n, and this set is irreducible.

Let A and B be free Abelian groups with respective bases $\{a_1,\ldots,a_k\}$ and $\{b_1,\ldots,b_m\}$, and let $C = A\times B$; T is a free ZA-module with basis $\{t_1,\ldots,t_n\}$; S is a free ZB-module with basis $\{s_1,\ldots,s_m\}$; $\overline{T}$ is a free ZC-module with basis $\{t_1,\ldots,t_n,s_1,\ldots,s_m\}$. Note that a subgroup of $M(B,S)$, generated by the matrices

$$x_1=\begin{pmatrix} b_1 & 0\\ s_1 & 1\end{pmatrix},\ \ldots,\ x_m=\begin{pmatrix} b_m & 0\\ s_m & 1\end{pmatrix},$$

is a free metabelian group F_m (see [6]).

Proposition 9. A subgroup of $M(C,\overline{T})$, generated by $M(A,T)$ and F_m, is equal to $M(A,T)\circ F_m$.

A consequence of Propositions 8 and 9 is the following:

COROLLARY. Assume that a u-group G embeds in $M(A,T)$, and either G is non-Abelian, or G lacks in non-identity unitriangular matrices, that is, $G\cap \mathrm{Fit}(M(A,T))=1$. Then a subgroup of $M(C,\overline{T})$, generated by G and F_m, is equal to $G\circ F_m$.

Proof of Proposition 9. Put $G=M(A,T)\circ F_m$ and denote by $\overline{G}$ the subgroup of $M(C,\overline{T})$ generated by $M(A,T)$ and F_m. Let $\varphi: G\to\overline{G}$ be a canonical epimorphism and $g\in\ker\varphi$. Then $g\in H=\mathrm{Fit}(G)=\mathrm{Fit}(M(A,T))\cdot[G,G]$. We treat $\mathrm{Fit}(G)$ as a ZC-module. By Proposition 4, such contains a submodule H_0, which, being a Z-module, decomposes into the direct sum

$$\sum_{b\in B}^{\oplus}\mathrm{Fit}(M(A,T))\cdot b\oplus\sum_{a\in A}^{\oplus}[F_m,F_m]\cdot a.$$

Notice that $\mathrm{Fit}(M(A,T))$ is identified with T, and $[F_m,F_m]$ — with some submodule of S. Since

$$\overline{T}=\sum_{b\in B}^{\oplus}T\cdot b\oplus\sum_{a\in A}^{\oplus}S\cdot a,$$

the restriction of φ to H_0 is an embedding. Therefore if $g\in H_0$ then $g=1$.

Let $g\notin H_0$. Then g represents a non-trivial element in the ZC-module H/H_0, which is isomorphic to a commutant of the metabelian product $A*B$ by Prop. 4. In turn the last-mentioned module, in view of Proposition 5, is isomorphic to the ideal $(A-1)(B-1)\cdot ZC$ of ZC, in which case the isomorphism is determined by a map $[a,b]\to(1-a)(1-b)$, $a\in A$, $b\in B$. Denote by σ a group homomorphism $M(C,\overline{T})\to M(C,ZC)$, which is induced by an identity group automorphism $C\to C$ and by a module homomorphism $\overline{T}\to ZC$, defined by the following map of the basis for $\overline{T}$:

$$t_i\to 0\ (i=1,\ldots,n),\quad s_j\to 1-b_j\ (j=1,\ldots,m).$$

Note that $M(A,T)\sigma \cong A$, $F_m\sigma \cong B$, and

$$\left[\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}, \quad x_j\right]\sigma = \begin{pmatrix} 1 & 0 \\ (1-a_i)(1-b_j) & 1 \end{pmatrix}.$$

Hence $M(C,\overline{T})\sigma \cong A * B$. We have arrived at a contradiction with the fact that the canonical image of g in $A*B$ is distinct from 1 on the one hand and $g\varphi = 1$ in $M(C,\overline{T})$ on the other. The proposition is proved.

Proposition 10. Let G be a non-Abelian u-group; then $G(x_1,\ldots,x_n)$ is G-discriminated by G.

Proof. Since the operation of taking u-products is associative, we can prove the statement for $n=1$, and then use induction. For $n=1$, the present proposition is valid in view of the following:

Proposition 11. Let $v(x)$ be a non-trivial element of $G(x)$, and let $L = \{g \in G \mid v(g) \neq 1\}$. We fix a tuple $\{g_1,\ldots,g_k\}$ of elements in G constituting a basis for G modulo $\mathrm{Fit}(G)$, and consider a set $S = \{g_1^{\alpha_1}\ldots g_k^{\alpha_k} \mid \alpha_i \in Z\}$. Then there exist a cofinite subset L_1 in S and a cofinite subset $L_2(s)$ in $\mathrm{Fit}(G)$, for every $s \in L_1$, such that $L \supseteq s \cdot L_1(s)$ for any $s \in L_1$.

Remark. The denotation $v(g)$ is correct. Indeed, if $v(x), w(x) \in G[x]$ represent a same element in $G(x)$, then $v(g) = w(g)$, for any $g \in G$, by the definition of a u-product.

Proof of Proposition 11. We assume that $G(x)$, in accordance with the corollary to Proposition 8, embeds in $M(C,\overline{T})$, so that $A = \langle a_1,\ldots,a_k\rangle \cong G/\mathrm{Fit}(G)$, $g_i \equiv a_i \bmod \mathrm{Fit}(M(C,\overline{T}))$ $(i=1,\ldots,k)$, $B = \langle b\rangle$, $C = A \times B$, T is a free ZA-module with basis $\{t_1,\ldots,t_n\}$, $\overline{T}$ is a free ZC-module with basis $\{t_1,\ldots,t_n,t\}$, $G \leqslant M(A,T)$, and $x = \begin{pmatrix} b & 0 \\ t & 1 \end{pmatrix}$.

First, let $v(x) \notin \mathrm{Fit}(M(C,\overline{T}))$. If $v(x) \equiv g_1^{\alpha_1}\ldots g_k^{\alpha_k}x^{\alpha} \bmod \mathrm{Fit}(M(C,\overline{T}))$ then $\alpha_1,\ldots,\alpha_k,\alpha$ are not all equal to zero. If α does not divide at least one of $\alpha_1,\ldots,\alpha_k$, then obviously $L = G$. But if $\alpha_1 = \beta_1\alpha,\ldots,\alpha_k = \beta_k\alpha$ $(\beta_i \in Z)$, then we put $L_1 = S \setminus \{g_1^{\beta_1}\ldots g_k^{\beta_k}\}$ and $L_2 = \mathrm{Fit}(G)$. We have $L \supseteq L_1L_2$.

Let

$$v(x) \in \mathrm{Fit}(M(C,\overline{T})) = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix},$$

$$v(x) = \begin{pmatrix} 1 & 0 \\ t_1u_1 + \ldots + t_nu_n + tu & 1 \end{pmatrix},$$

where

$$u_1 = u_1(b),\ \ldots,\ u_n = u_n(b),\ u = u(b) \in ZC = ZA[B].$$

Denote by $\overline{g}$ the image of an element $g \in G$ in the group A. By the hypothesis, $u_1(b),\ldots,u_n(b),u(b)$ are not all equal to zero. Hence there exists a cofinite subset L_1 in S such that, for $s \in L_1$, the non-zero elements go to non-zero whenever s is substituted for b in $u_1(b),\ldots,u_n(b),u(b)$. If $u(b)=0$ then we put $L_2 = \mathrm{Fit}(G)$. Therefore $v(g) \neq 1$ for any $g \in L_1L_2$.

Suppose $u(b) \neq 0$ and $h = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ runs over elements of the group $\mathrm{Fit}(G)$, which is non-trivial by assumption. Let $s = \begin{pmatrix} \overline{s} & 0 \\ s_0 & 1 \end{pmatrix} \in L_1$. Substituting the matrix $sh = \begin{pmatrix} \overline{s} & 0 \\ s_0 + y & 1 \end{pmatrix} \in L_1$ for x in $v(x)$ yields a matrix hosting the expression $t_1u_1(\overline{s}) + \ldots + t_nu_n(\overline{s}) + yu(\overline{s})$ in its lower left corner. Evidently, $\overline{T}$ contains at most one value of y for which the expression in question is equal to zero. Therefore there exists at most one element $h \in \mathrm{Fit}(G)$ such that $v(sh) = 1$. Dropping a suitable element from $\mathrm{Fit}(G)$ (if any), we arrive at the desired set $L_2(s)$. The proposition is proved.

The **proof** of Theorem 1 now follows immediately from Propositions 1, 10 and the corollary to Props. 8, 9.

4. IRREDUCIBLE ALGEBRAIC SETS IN G

4.1. Let G be a non-Abelian u-group, S be an irreducible algebraic set in G, and Γ be the coordinate group of S. Theorem 1 implies that Γ is a factor group of $G(x) = G \circ X$, where $X = \langle x \rangle$. By Proposition 1, Γ is G-universally equivalent to G and, specifically, is a u-group. The case where $\Gamma = G$ checks with S being a singleton or $\chi(\Gamma) = (0,0)$, and $\Gamma = G(x)$ means that $S = G$ or $\chi(\Gamma) = (1,1)$.

A singleton, or the whole group G, is referred to as an *irreducible set of type* 1. We omit this type from further consideration, assuming that S is an infinite set, strictly less than G. In this event the image of x in Γ is other than 1; so, X embeds in Γ. There are two versions: $X \leqslant \mathrm{Fit}(\Gamma)$ or $X \cap \mathrm{Fit}(\Gamma) = 1$. These check with the conditions that $\chi(\Gamma) = (0,1)$ and $\chi(\Gamma) = (1,0)$.

4.2. Let $A = G/\mathrm{Fit}(G)$. Recall that $\mathrm{Fit}(G)$ may be conceived of as a right ZA-module. Denote by $G \oplus X$ the group obtained from G by adding to the module $\mathrm{Fit}(G)$ a direct summand, a free one-generated ZA-module $x \cdot ZA$.

Proposition 12. Under the above assumptions on G, S, and Γ, $X \leqslant \mathrm{Fit}(\Gamma)$. Then $\Gamma = G \oplus X$ and $S = \mathrm{Fit}(G)$.

Proof. The condition $X \leqslant \mathrm{Fit}(\Gamma)$ readily implies that Γ is a factor group of $\Gamma = G \oplus X$. It is also clear that substituting any element $f \in \mathrm{Fit}(G)$ for x determines a G-epimorphism $\varphi_f : G \oplus X \to G$. Such epimorphisms, note, discriminate the group $G \oplus X$. The latter follows from the observation that if a non-unit element h of $G \oplus X$ does not lie in $\mathrm{Fit}(G \oplus X)$ then its image, under any epimorphism φ_f, is other than 1, but if $h \in \mathrm{Fit}(G \oplus X)$ then h, in the additive notation, can be written in the form $xu + v$, where $u \in ZA$, $v \in \mathrm{Fit}(G)$, and there exists at most one element $f \in \mathrm{Fit}(G)$ such that $h\varphi_f = fu + v = 0$.

By Proposition 1, $G \oplus X$ is the coordinate group of some irreducible algebraic set in G. The fact that $\chi(\Gamma) = \chi(G \oplus X) = (0,1)$ and Proposition 3 imply that $\Gamma = G \oplus X$. Further, we note that $G \oplus X$ appears as the factor group of $G[x]$ w.r.t. the normal closure of a set $\{[x,g] \mid g \in \mathrm{Fit}(G)\}$. On the other hand, the set of solutions for all, or even one, of the equations $[x,g] = 1$ $(1 \neq g \in \mathrm{Fit}(G))$ coincides with $\mathrm{Fit}(G)$. Hence $S = \mathrm{Fit}(G)$, and the proposition is thus proved.

Every coset w.r.t. $\mathrm{Fit}(G)$ is called an *irreducible set of type* 2.

4.3. Let $X \cap \mathrm{Fit}(G) = 1$. Any equation solved by $x = g$ can be written in the form $(xg^{-1})^{\alpha} v(x) = 1$, where $\alpha \in Z$ and $v(x) \in \mathrm{Fit}(G(x))$. If $\alpha \neq 0$, then we substitute $x = yg$ to see that solutions of the equation in y should lie in $\mathrm{Fit}(G)$. We are so faced up to the case treated in the previous section, and can assert that $S = \mathrm{Fit}(G) \cdot g$.

Thus we are left to consider the main case where S is not a set of type 1 or 2 and satisfies a non-trivial equation $v(x) = 1$, where $v(x) \in \mathrm{Fit}(G(x))$. Assume that the group $G(x)$ is embedded in $M(C, \overline{T})$, where $C = A \times B$, $A = \langle a_1, \ldots, a_k \rangle \cong G/\mathrm{Fit}(G)$, $B = \langle b \rangle$, $\overline{T}$ is a free ZC-module with basis $\{t_1, \ldots, t_n, t\}$, and $x = \begin{pmatrix} b & 0 \\ t & 1 \end{pmatrix}$. Then $v(x) = 1$ is equivalent to some equation in $\overline{T}$ in b and t, that is,

$$t_1 u_1(b) + \ldots + t_n u_n(b) + t u(b) = 0, \tag{1}$$

in which case values for b are searched for in the group A, and those for t — in the module $T = t_1 ZA + \ldots + t_n ZA$. In addition, substituting corresponding values into $\begin{pmatrix} b & 0 \\ t & 1 \end{pmatrix}$ should yield an element of G. We look at solutions for equation (1) in just this sense. Note from the outset that $u(b) \neq 0$ in our equation, for otherwise it reduces to the system $u_1(b) = 0, \ldots, u_n(b) = 0$. In this instance the solution for the initial system (if it is not empty) is a union of several cosets w.r.t. $\mathrm{Fit}(G)$, which contradicts the assumption on S. Without loss of generality, further we can suppose that $u(b)$ is a polynomial in b with coefficients from ZA and that its constant term is distinct from zero.

In what follows, we limit ourselves to treating two types of G. We assume that either $G = M(A,T)$ or $G = F_n = \langle x_1, \ldots, x_n \rangle$, and so $k = n$ and $x_1 = \begin{pmatrix} a_1 & 0 \\ t_1 & 1 \end{pmatrix}, \ldots, x_n = \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix}$.

We need the following:

LEMMA 1 [6]. Let $G = F_n$. The matrix

$$\begin{pmatrix} a & 0 \\ t_1 v_1 + \ldots t_n v_n & 1 \end{pmatrix} \in M(A,T)$$

lies in G if and only if $(a_1 - 1)v_1 + \ldots + (a_n - 1)v_n = a - 1$.

For the case where $G = M(A,T)$, the left part of equation (1) may be arbitrary, and if $G = F_n$, then

$$(a_1 - 1)u_1(b) + \ldots + (a_n - 1)u_n(b) + (b - 1)u(b) = 0 \tag{2}$$

should hold by Lemma 1 and in view of

$$\begin{pmatrix} 1 & 0 \\ t_1 u_1 + \ldots t_n u_n + tu & 1 \end{pmatrix} \in F_{n+1} = \langle x_1, \ldots, x_n, x \rangle.$$

Obviously condition (2) remains true if $u_1(b), \ldots, u_n(b), u(b)$ is cancelled by a common factor. Let $d(b)$ be the greatest common divisor of $u_1(b), \ldots, u_n(b), u(b)$ in ZC. Then equation (1) is equivalent to

$$d(b) = 0,\ t_1 \cdot u_1(b)d(b)^{-1} + \ldots + t_n \cdot u_n(b)d(b)^{-1} + t \cdot u(b)d(b)^{-1} = 0.$$

If the first equation of the above system is soluble then it defines a union of some cosets of G w.r.t. $\mathrm{Fit}(G)$. Since S is not a set of type 1 or 2, it should satisfy the second equation in the system. In other words, we may assume that (1) satisfies the following condition:

$$\text{the greatest common divisor of } u_1(b), \ldots, u_n(b), u(b) \text{ is equal to unity.} \tag{$*$}$$

LEMMA 2. The set S is finite modulo $\mathrm{Fit}(G)$ (under the assumption that S is not a set of type 1 or 2).

Proof. Assuming the contrary, we write S in the form $g_1 \cdot S_1 \cup \ldots \cup g_m \cdot S_m$, where $g_1, \ldots, g_m \in G$ are distinct cosets w.r.t. $\mathrm{Fit}(G)$ and $S_1, \ldots, S_m$ are subsets in $\mathrm{Fit}(G)$. Then every set $g_i \cdot S_i$, being an intersection of S and $g_i \cdot \mathrm{Fit}(G)$, is algebraic. The set S being irreducible implies that $m = 1$ and S must be either a singleton or a coset w.r.t. $\mathrm{Fit}(G)$, that is, S should be of types 1 or 2, which it is not by assumption. The lemma is proved.

LEMMA 3. Under $(*)$, the element $u(b)$ does not in fact depend on b.

In order to prove the statement, we need two instrumental lemmas.

LEMMA 4. Let K be a field, and let $f(x,y)$ and $g(x,y)$ be elements of a ring $K[x, x^{-1}, y, y^{-1}]$. Then there exists a natural number $n_0 = n_0(f,g)$ such that if $n > n_0$, $0 \neq \xi \in K$, then the condition that $f(x, \xi x^n)$ divides $g(x, \xi x^n)$ in a ring $K[x, x^{-1}]$ implies that $f(x,y)$ divides $g(x,y)$ over a field $K(x)$.

Proof. Multiplying elements $f(x,y)$ and $g(x,y)$ by suitable degrees of x and y, we can reduce our problem to the case where these elements lie in $K[x,y]$, and are not divisible by x and y in that ring. Suppose the constant term in y of a polynomial $f(x,y)$ is divisible by x^l, where l is maximal and $l > 0$. Then we substitute $y \to yx^l$ and divide $f(x,y)$ by x^l. Eventually, we obtain a polynomial whose constant term in y is not divisible by x.

It suffices to consider a pair of transformed polynomials. Therefore we assume that $f(x,y)$ itself possesses this property. Clearly, for any natural n, then, the polynomial $f(x,\xi x^n)$ is not divisible by x, and if it divides $g(x,\xi x^n)$ in the ring $K[x,x^{-1}]$, then the fraction of division lies in $K[x]$. Let m be the degree of $f(x,y)$ w.r.t. y. We divide g by f with remainder, that is, consider a representation $g(x,y)\alpha(x) = f(x,y)h(x,y) + r(x,y)$, where $\alpha(x) \in K[x]$, $h(x,y), r(x,y) \in K[x,y]$, and the degree of $r(x,y)$ w.r.t. y does not exceed $m-1$. As n_0 we take a maximum of degrees of $f(x,y)$ and $r(x,y)$ w.r.t. x. Let $n > n_0$, $0 \neq \xi \in K$, and $f(x,\xi x^n)$ divide $g(x,\xi x^n)$ in the ring $K[x,x^{-1}]$, and hence also in $K[x]$. Note that the degree of $f(x,\xi x^n)$ w.r.t. x is at least mn, and the degree of $r(x,\xi x^n)$ w.r.t. x is strictly less than mn. The condition of divisibility implies that $r(x,\xi x^n) = 0$, and so $r(x,y) = 0$. The lemma is proved.

Obviously, we have

LEMMA 5. Let K be a commutative integral domain with unique decomposition, K' be its field of fractions, and $f(x)$ be an irreducible element of a ring $K[x,x^{-1}]$, where $f(x)$ is not associated with an element of K. Then $f(x)$ is indecomposable in the ring $K'[x,x^{-1}]$, and if $g(x) \in K[x,x^{-1}]$ and $f(x)$ divides $g(x)$ in $K'[x,x^{-1}]$, then the fraction of division lies in $K[x,x^{-1}]$.

Proof of Lemma 3. Suppose, on the contrary, that $u(b)$ depends on b. Recall that G is embedded in the matrix group $M(A,T)$. Denote by D the projection of a set S onto A. By Lemma 2, the set D is infinite. There is no loss of generality in assuming that D contains an infinite subset $L = \{a_1^{n_i}\xi_i \mid i \in N\}$, where n_i are distinct natural numbers, with $\xi_i \in \langle a_2,\ldots,a_k\rangle$. Denote by K the field of fractions of a ring $Z\langle a_2,\ldots,a_k\rangle$.

If $\begin{pmatrix} a & 0 \\ w & 1 \end{pmatrix} \in S$ then $b = a, t = w$, by assumption, is a solution for equation (1). In particular, we may assert that for any $a \in L$, the element $u(a)$ divides $u_1(a),\ldots,u_n(a)$ in the ring $K[a_1,a_1^{-1}]$. By Lemma 4, $u(b)$ divides $u_1(b),\ldots,u_n(b)$ in a ring of polynomials in b over a field of fractions for ZA, and by Lemma 5, any prime element of ZC that divides $u(b)$ and is not associated with an element of ZA will divide $u_1(b),\ldots,u_n(b)$ in ZC. By $(*)$, the greatest common divisor of $u_1(b),\ldots,u_n(b),u(b)$ in ZC is equal to 1. Hence all prime divisors of $u(b)$ should be associated with elements of ZA, and since $u(b)$ is a polynomial in b over ZA, whose constant term is other than zero, we have $u(b) \in ZA$. The lemma is proved.

LEMMA 6. A set S can satisfy only one (up to colinearity) equation of the form (1).

Otherwise, two equations may be used to obtain the third one in form (1), for which $u(b) = 0$.

LEMMA 7. The canonical mapping $S \to G/\mathrm{Fit}(G) = A$ is an injection.

Proof. Let $\begin{pmatrix} a & 0 \\ w_1 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ w_2 & 1 \end{pmatrix} \in S$. Since $b = a, t = w_1$ and $b = a, t = w_2$ are solutions for equation (1), we have $(w_1 - w_2)u(a) = 0$, and since $u(b)$ does not indeed depend on b and is distinct from zero, we obtain $w_1 = w_2$. The lemma is proved.

4.4. Consider the case where $u = 1$, that is, S satisfies the equation

$$t_1u_1(b) + \ldots + t_nu_n(b) + t = 0. \tag{3}$$

For $G = F_n$, this case is impossible, since $\mathrm{Fit}(G(x)) = [G(x), G(x)]$ in this instance, and so all elements $u_1(b), \ldots, u_n(b), u(b)$ should belong to the difference ideal of a group ring ZC. We therefore assume that $G = M(A, T)$. Then any equation of the form (3) defines an algebraic set S, which, by Lemma 6, is irreducible and is injectively projected onto the group, that is, $S \to G/\mathrm{Fit}(G) = A$. We say that such a set has *type* 3.

In virtue of the fact that equation (3) properly defines S, its coordinate group can be identified with a G-subgroup of $M(C, \overline{T})$, generated by the element

$$\begin{pmatrix} b & 0 \\ -t_1u_1 - \ldots - t_nu_n & 1 \end{pmatrix},$$

which is obtained from x by replacing t by $-t_1u_1 - \ldots - t_nu_n$.

4.5. Further, we consider an irreducible algebraic set S of *type* 4, which is defined by the equation in form (1), where u is an invertible element of ZA. Of course such an equation cannot be arbitrary.

LEMMA 8. The matrix $\begin{pmatrix} d & 0 \\ w & 1 \end{pmatrix}$ in $M(A, T)$, where $w = t_1w_1 + \ldots + t_nw_n$, lies in S if and only if u divides all elements $u_1(d), \ldots, u_n(d)$ in ZA, and moreover, $w_1 = -u_1(d)u^{-1}, \ldots, w_n = -u_n(d)u^{-1}$.

Proof. If $\begin{pmatrix} d & 0 \\ w & 1 \end{pmatrix} \in S$ then $b = d, t = w$ is a solution for equation (1), whence $w_1 = -u_1(d)u^{-1}$, $\ldots$, $w_n = -u_n(d)u^{-1}$. To argue for the way back, let $w_1 = -u_1(d)u^{-1}$, $\ldots$, $w_n = -u_n(d)u^{-1} \in ZA$. Then $b = d, t = w$ is a solution for (1). Also it is necessary that $\begin{pmatrix} d & 0 \\ w & 1 \end{pmatrix}$ lie in G. This is obviously so if $G = M(A, T)$. For $G = F_n$, condition (2) implies $(a_1 - 1)w_1 + \ldots + (a_n - 1)w_n = d - 1$. By Lemma 1, this does mean that $\begin{pmatrix} d & 0 \\ w & 1 \end{pmatrix}$ lies in G. The lemma is proved.

For a given set S, denote by D its projection onto A. Our next goal is to study the structure of a set D. Let $\pi = \pi(u)$ be the set of all prime elements of ZA dividing u, and let $u = \prod_{p \in \pi} p^{\alpha(p)}$. We may assume that elements of π lie in the polynomial ring $Z[a_1, \ldots, a_k]$ and have non-trivial constant terms. By Lemma 8, D consists of exactly those elements $d \in A$ for which u divides $u_1(d), \ldots, u_n(d)$ in ZA. Denote by $D(p)$ $(p \in \pi)$ the set of elements $d \in A$ for which $p^{\alpha(p)}$ divides $u_1(d), \ldots, u_n(d)$. Then $D = \bigcap_{p \in \pi} D(p)$. We look at sets $D(p)$ $(p \in \pi)$.

LEMMA 9. Let K be a commutative integral domain with unique decomposition, $f(x) \in K[x]$, n be the degree of a polynomial $f(x)$, $\{d_0, \ldots, d_n\}$ be a tuple of distinct elements in K, and p be a prime element of the ring K such that p divides $f(d_i)$ but does not divide $d_i - d_j$, for all $i \neq j$. Then p divides $f(x)$.

Proof. Let $\overline{K}$ be a field of fractions for the factor ring K/pK, and let $\overline{d}$ be the image of an element $d \in K$ in $\overline{K}$ and $\overline{f}(x)$ be the image of a polynomial $f(x)$ in $\overline{K}[x]$. The hypotheses of the lemma imply that the polynomial $\overline{f}(x)$, whose degree does not exceed n, has $n+1$ distinct roots $\overline{d}_0, \ldots, \overline{d}_n$ in $\overline{K}$. Hence $\overline{f}(x) \equiv 0$, and so p divides $f(x)$. The lemma is proved.

For any non-zero element $v(b) \in ZC$, denote by $\widetilde{v}(b)$ the element $b^{-l}v(b)$, where b^l is a minimal degree of b occurring in the decomposition of $v(b)$. Clearly, $\widetilde{v}(b)$ is a polynomial in b with coefficients from ZA and non-zero constant term.

Let $p \in \pi$. Since p does not divide one of the polynomials $\widetilde{u}_i(b)$ but divides all of its values $\widetilde{u}_i(d)$ whenever $d \in D(p)$, it follows by Lemma 8 that the set $D(p)$ splits into a finite (not exceeding the degree of $\widetilde{u}_i(b)$) number of classes of elements that are comparable modulo p. Let $D_0(p)$ be the union of classes containing finitely many elements and $D'(p)$ be the union of classes containing infinitely many. Suppose that $d_1, \ldots, d_m$ are representatives of all classes in $D'(p)$.

If, for instance, d_1, d, d' lie in one class, then p divides $1-d_1^{-1}d = d_1^{-1}(d_1-d)$ and $1-d_1^{-1}d' = d_1^{-1}(d_1-d')$. Specifically, the elements $1-d_1^{-1}d$ and $1-d_1^{-1}d'$ are not coprime in ZA. Therefore $d_1^{-1}d$ and $d_1^{-1}d'$ sit in one cyclic subgroup of A, and elements of the class defined by d_1 lie in one coset w.r.t. that subgroup. We may assume that the subgroup appearing is isolated in A. Let it be equal to $\langle a \rangle$. Then p divides some element a^l-1, $l \in N$. Note that prime divisors of a^l-1 in ZA are exactly integral polynomials in a that are minimal, over the field of rationals, for roots of unity of degrees dividing l. Thus p is a minimal polynomial for the primitive root of unity of some degree $r = r(p)$. We know that p divides a^l-1 iff r divides l. Put $A(p) = \langle a^{r(p)} \rangle$. Then

$$D(p) = D_0(p) \cup D'(p), \; D'(p) \subseteq d_1 \cdot A(p) \cup \ldots \cup d_m \cdot A(p). \tag{4}$$

LEMMA 10. We have

$$D'(p) = d_1 \cdot A(p) \cup \ldots \cup d_m \cdot A(p). \tag{5}$$

Proof. It suffices to show, for instance, that $d_1 \cdot A(p) \subseteq D(p)$. There is no loss of generality in assuming $d_1 = 1$. We have the following: the group $A(p) = \langle a^r \rangle$ contains an infinite set of elements d such that p^α divides $\widetilde{u}_i(d)$. We need to verify that p^α divides $\widetilde{u}_i(d)$, for all $d \in A(p)$. This fact will be derived from Lemma 11.

For any non-zero element $v \in ZA$, we put $\omega(v) = \max\{\beta \mid p^\beta \text{ divides } v\}$, with $\omega(0) = \infty$. The function ω is a valuation on ZA. Note also that $\omega(d-1) = 1$, for any non-unit element $d \in A(p)$.

LEMMA 11. Let $0 \neq f(x) \in ZA[x]$. We expand the polynomial $f(x)$ in terms of degrees of $x-1$ as follows: $f(x) = v_0 + v_1(x-1) + \ldots + v_n(x-1)^n$. Let $k = \min\{\omega(v_i) + i \mid i = 0, \ldots, n\}$ and $\{m_0, \ldots, m_k\}$ be a tuple of different integers. Then $\min\{\omega(f(a^{rm_i})) \mid i = 0, \ldots, k\} = k$.

Proof. Put $c = a^r$ and $f_0 = f(c^{m_0}), \ldots, f_k = f(c^{m_k})$. It is clear that $\omega(f(c^m)) \geqslant k$, for any integer m. Therefore we need to prove that $\omega(f_i) = k$, for some index i. We may assume that the degree of $f(x)$ w.r.t. $x-1$ does not exceed $k+1$, since summands with higher degrees for $x = c^m$ constitute an element whose value is at least $k+1$. We can also suppose that p does not divide any one of $v_0, \ldots, v_k$; otherwise, we may replace f by fp^{-1} and use induction on k. We have the following system of equalities:

$$\begin{cases} v_0 + v_1(c^{m_0}-1) + \ldots + v_k(c^{m_0}-1)^k = f_0 \\ \ldots \\ v_0 + v_1(c^{m_k}-1) + \ldots + v_k(c^{m_k}-1)^k = f_k, \end{cases}$$

which may be conceived of as a system of equations in $v_0, \ldots, v_k$. The determinant Δ of this system is the Vandermonde determinant, equal to $\prod\limits_{i<j}(c^{m_i} - c^{m_j})$. Since $\omega(c^{m_i} - c^{m_j}) = 1$, it follows that $\omega(\Delta) = k(k+1)/2$. By a classical formula, $v_i = \Delta_i/\Delta$, where Δ_i is obtained from Δ by replacing the ith column by a column of constant terms. It is easy to see that $\omega(\Delta_i) > \omega(\Delta)$ if $\omega(f_0) \geqslant k+1, \ldots, \omega(f_k) \geqslant k+1$, and so each element v_i is divisible by p, which contradicts the hypothesis. The lemma is proved.

We come back to the proof of Lemma 10. Lemma 11 implies that $\widetilde{u}_i(b)$ is representable as $v_0 + v_1(b-1) + \ldots + v_k(b-1)^k$, where $v_j \in ZA$ and $\omega(v_j) \geqslant \alpha - j$ $(j = 0, \ldots, k)$. But then $\omega(\widetilde{u}_i(d)) \geqslant \alpha$, for any $d \in A(p)$, that is, p^α divides $\widetilde{u}_i(d)$, proving Lemma 10.

LEMMA 12. All elements of π are divisors of a^l-1, for some natural l.

Proof. Let $q \in \pi$, $q \neq p$. Consider an expansion of the set $D(q)$, similar to one obtained for $D(p)$, that is, $D(q) = D_0(q) \cup D'(q)$, $D'(q) = e_1 \cdot A(q) \cup \ldots \cup e_f \cdot A(q)$. In view of the fact that the set $D(p) \cap D(q)$ contains D and, in particular, is infinite, one of the sets $d_i \cdot A(p) \cap e_j \cdot A(q)$, too, is infinite. Consequently, the cyclic subgroups $A(p)$ and $A(q)$ have a non-trivial intersection, and so $\langle a \rangle \supseteq A(q)$ and q divides some element $a^l - 1$. The lemma is proved.

Put $A' = \bigcap\limits_{p \in \pi} A(p) = \langle a' \rangle$. By construction, any element $p \in \pi$ divides $a' - 1$. Recall that $D = \bigcap\limits_{p \in \pi} D(p)$, and hence $D = D_0 \cup D'$, where D_0 is a finite set, and $D' = \bigcap\limits_{p \in \pi} D'(p)$. For every $p \in \pi$, we take one coset $d(p) \cdot A(p)$ occurring in the expansion of $D'(p)$. Assume that the cosets chosen have a non-empty intersection. Then this intersection, together with each element h, contains the class hA' as a whole and, consequently, splits into a union of several cosets w.r.t. A'. Hence the set D' is the union of finitely many cosets w.r.t. A'.

Note also that the coordinate group of the set S in question can be identified with a G-subgroup of $M(C, \overline{T}u^{-1})$, generated by the matrix

$$\begin{pmatrix} b & 0 \\ -t_1u_1u^{-1} - \ldots - t_nu_nu^{-1} & 1 \end{pmatrix},$$

which is obtained from x by replacing t by $-t_1u_1u^{-1} - \ldots - t_nu_nu^{-1}$.

We sum up the basic facts proved in this section.

THEOREM 2. Let the non-commutative metabelian group G be equal to $M(A, T) = W_{nk}$, or to a free metabelian group F_n, and let S be an irreducible algebraic subset of G and D be its image in $A = G/\mathrm{Fit}(G)$. Then one of the following holds:

(1) $S = \{g\}$ is a singleton;

(2) $S = G$;

(3) $S = g \cdot \mathrm{Fit}(G)$ is a coset w.r.t. the Fitting subgroup;

(4) the map $S \to D$ is bijective, and $D = A$ (which is possible only for $G = M(A, T)$);

(5) the map $S \to D$ is bijective, and $D = D_0 \sqcup d_1 \cdot A' \sqcup \ldots \sqcup d_m \cdot A'$, where D_0 is a finite set and A' is a cyclic subgroup of A.

4.6. In conclusion, we furnish an example showing that Theorem 2(5) is realized in the general form for $G = M(A, T)$. Suppose the infinite subset D in A splits into the union $D_0 \sqcup d_1 \cdot A' \sqcup \ldots \sqcup d_m \cdot A'$, where $D_0 = \{c_1, \ldots, c_l\}$ is a finite set, $A' = \langle a^r \rangle$ is a cyclic subgroup of A, and the element a is not a proper degree of some other element in A. Denote by $p = p(a)$ the minimal polynomial in a for the primitive root of unity of degree r. Put

$$v(b) = \prod_{i=1}^{l} (c_i^{-1}b - 1) \cdot \prod_{j=1}^{m} (d_j^{-1}b - 1)^{l+1}.$$

Consider an algebraic subset S in $G = M(A, T)$, defined by the equation

$$t_1(p^{l+1} + v(b)) = tp^{l+1}, \tag{6}$$

and denote by $D(S)$ the projection of S onto A. Since p^{l+1} divides $p^{l+1} + v(d)$ for $d \in D$, it follows that $D \subseteq D(S)$. We argue to state the inverse inclusion. Let $c \in A$ and p^{l+1} divide $p^{l+1} + v(c)$. Then p divides one of $c_i^{-1}c - 1$, $d_j^{-1}c - 1$. If p divides $d_j^{-1}c - 1$ then $c \in d_j \cdot A' \subseteq D$. For instance, let p divide $c_1^{-1}c - 1$. This is equivalent to $c \in c_1 \cdot A'$. Since the coset $c_1 \cdot A'$ is distinct from each one of the classes $d_1 \cdot A', \ldots, d_m \cdot A'$,

p does not divide $\prod_{j=1}^{m}(d_j^{-1}c-1)^{l+1}$. Hence p^{l+1} divides $\prod_{i=1}^{l}(c_i^{-1}c-1)$. If $c_i^{-1}c-1$ is distinct from zero then p^2 does not divide this element; so, there is zero among the factors $c_i^{-1}c-1$, whence $c \in D_0$.

Thus the projection of S onto A coincides properly with D. And the algebraic set S, note, is irreducible since the coefficients in (6) defining S are coprime.

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