

FULLY RESIDUALLY FREE GROUPS AND GRAPHS LABELED BY INFINITE WORDS

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Let $F = F(X)$ be a free group with basis X and $\mathbb{Z}[t]$ be a ring of polynomials with integer coefficients in t . In this paper we develop a theory of $(\mathbb{Z}[t], X)$ -graphs — a powerful tool in studying finitely generated fully residually free (limit) groups. This theory is based on the Kharlampovich–Myasnikov characterization of finitely generated fully residually free groups as subgroups of the Lyndon’s group $F^{\mathbb{Z}[t]}$, the author’s representation of elements of $F^{\mathbb{Z}[t]}$ by infinite $(\mathbb{Z}[t], X)$ -words, and Stallings folding method for subgroups of free groups. As an application, we solve the membership problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$, as well as for finitely generated fully residually free groups.

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1. Introduction

In the seminal paper [12] Stallings introduced an extremely useful notion of a *folding* of graphs and initiated the study of subgroups (and automorphisms) of free groups via folded directed labeled graphs. This approach turned out to be very influential and allowed researchers to prove many new results and simplify old proofs. We refer

to a survey [4] for a detailed discussion of the known results and methods in this area. Here we would like only to mention that, among other things, the Stallings technique gives a very fast algorithm for the *Uniform Membership Problem* in free groups. Namely, given a tuple of words h_1, \dots, h_n of a free group $F(X)$ with basis X one can construct in at most quadratic time in $|h_1| + \dots + |h_n|$ the Stallings folding (viewed as a deterministic finite automata) Γ_H which accepts precisely the reduced words from $F(X)$ which belong to the subgroup $H = \langle h_1, \dots, h_n \rangle$. Now, given a reduced word $w \in F(X)$ one can check, in linear time in $|w|$, whether $w \in H$ or not. One of the crucial features of the Stallings method is that the graph $\Gamma(H)$ carries all the essential information about the subgroup H itself.

Observe that free groups are precisely the groups that admit free actions on simplicial trees. In the paper [13] Stallings offered a way to extend his ideas to non-free actions of groups on graphs and trees. Bestvina and Feighn [2] and, independently, Dunwoody [3] gave a systematic treatment of Stallings's method in the context of graphs of groups and groups acting on simplicial trees. A more geometric and unified approach to this topic which relies on Bass–Serre theory [11, 1] as well as the foldings technique of Stallings–Bestvina–Feighn–Dunwoody was recently developed in [5].

Research on the membership problem in amalgamated free products, HNN-extensions and more generally, fundamental groups of graphs of groups has a long and rich history. In 1958 Mihailova [8] constructed a remarkable example of a finitely generated subgroup of the direct product $F_2 \times F_2$ of two free groups of rank two with unsolvable membership problem. Most of the early results used normal forms techniques, so the proofs were very technical and sometimes cumbersome. Bass–Serre theory gives a nice tool to deal with subgroups of fundamental groups of graphs of groups, but the algorithmic side of the theory is still underdeveloped. It seems, the first general attempt to provide an algorithmic version of Bass–Serre subgroup theorem via foldings was made [5]. We refer to this paper for a brief survey on the membership problem in fundamental groups of graphs of groups.

In the present paper, we develop folded graph methods for finitely generated subgroups H of finitely generated fully residually free groups G . Notice that such groups G are subgroups of groups obtained from a free group by a finite sequence of extensions of centralizers, so the methods of Bass–Serre theory, in particular, the results from [5], can be in use. However, we elect to generalize directly the original Stallings method for free groups on fully residually free ones. Namely, using Kharlampovich–Myasnikov embedding theorem [6] we embed the group G into the Lyndon's free $\mathbb{Z}[t]$ -group $F^{\mathbb{Z}[t]}$. Notice that such embedding can be found effectively. This allows one to view the groups G as subgroups of $F^{\mathbb{Z}[t]}$ given by a finite set of generators. Now, for a finitely generated subgroup H of $F^{\mathbb{Z}[t]}$ we effectively construct a folded labeled graph Γ which accepts precisely those elements of $F^{\mathbb{Z}[t]}$ (written in a normal form) which belong to H . To construct Γ we use the representation of elements from $F^{\mathbb{Z}[t]}$ by infinite words in the alphabet X (here X is a basis of F) that was introduced in [10]. Thus, we may assume that the subgroup H is generated by a finite set of infinite words. In this event, to build Γ we mimic the

classical Stallings folding algorithm replacing finite words by the infinite ones. This is not immediate, however, and it requires some new ideas. When Γ is constructed we solve the membership problem for H precisely in the same way as in free groups. In the subsequent paper we use these graphs Γ associated with the subgroups H of fully residually free groups to solve other algorithmic problems for H .

2. $F^{\mathbb{Z}[t]}$ and Its Elements Viewed as Infinite Words

In this section, at first, we review the embedding of $F^{\mathbb{Z}[t]}$ into the set of reduced infinite words. Then we describe normal forms for elements of $F^{\mathbb{Z}[t]}$ which arise from the embedding.

2.1. Lyndon’s free $\mathbb{Z}[t]$ -group and infinite words

Let $F = F(X)$ be a free non-abelian group with basis $X = \{x_i \mid i \in I\}$ and $\mathbb{Z}[t]$ be a ring of polynomials with integer coefficients in t . Let $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^\pm = X \cup X^{-1}$. In [7], Lyndon introduced a $\mathbb{Z}[t]$ -completion $F^{\mathbb{Z}[t]}$ of F , which it is called now Lyndon’s free $\mathbb{Z}[t]$ -group. It turns out that $F^{\mathbb{Z}[t]}$ can be described as a union of a sequence of extensions of centralizers (see [9])

$$F = G_0 < G_1 < \dots < G_n < \dots, \tag{2.1}$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i by a free abelian group of countable rank.

It was shown in [10] that elements of $F^{\mathbb{Z}[t]}$ can be viewed as *infinite words* defined in the following way. Let A be a discretely ordered abelian group. By 1_A we denote the minimal positive element of A . Recall that if $a, b \in A$ then the closed segment $[a, b]$ is defined as $[a, b] = \{x \in A \mid a \leq x \leq b\}$. An A -word is a function of the type

$$w : [1_A, \alpha_w] \rightarrow X^\pm,$$

where $\alpha_w \in A, \alpha_w \geq 0$. The element α_w is called the *length* $|w|$ of w . By ε we denote the empty word. We say that w is *reduced* if $w(\alpha) \neq w(\alpha+1)^{-1}$ for any $1_A \leq \alpha < \alpha_w$. Then, as in a free group, one can introduce a partial multiplication $*$, an inversion, a word reduction etc. on the set of all A -words (infinite words) $W(A, X)$. We write $u \circ v$ instead of uv if $|uv| = |u| + |v|$. All these definitions make it possible to develop infinite words techniques, which provide a very convenient combinatorial tool (for all the details we refer to [10]).

It was proved in [10] that $F^{\mathbb{Z}[t]}$ can be canonically embedded into the set of reduced infinite words $R(\mathbb{Z}[t], X)$, where $\mathbb{Z}[t]$, an additive group of polynomials with integer coefficients, is viewed as an ordered abelian group with respect to the standard lexicographic order \leq (that is, the order which compares the degrees of polynomials first, and if the degrees are equal, compares the coefficients of corresponding terms starting with the terms of highest degree). More precisely, the embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ was constructed by induction, that is, all G_i

from the series (2.1) were embedded step by step in the following way. Suppose, the embedding of G_i into $R(\mathbb{Z}[t], X)$ is already constructed. Then, one chooses a Lyndon's set $U_i \subset G_i$ (see [10]) and the extension of centralizers of all elements from U_i produces G_{i+1} , which is now also naturally embedded into $R(\mathbb{Z}[t], X)$.

The existence of an embedding of $F^{\mathbb{Z}[t]}$ into the set of infinite words implies automatically the fact that all subgroups of $F^{\mathbb{Z}[t]}$ are also subsets of $R(\mathbb{Z}[t], X)$, that is, their elements can be viewed as infinite words. From now on we assume the embedding $\rho : F^{\mathbb{Z}[t]} \rightarrow R(\mathbb{Z}[t], X)$ to be fixed. Moreover, for simplicity we identify $F^{\mathbb{Z}[t]}$ with its image $\rho(F^{\mathbb{Z}[t]})$.

Further we introduce an order on each U_i . Let $v \in U_i$. There exists an infinite subset $K_i(v) \subset U_i$ such that $u \in K_i(v)$ only if $|u| = |v|$. In fact, $K_i(u) = K_i(v)$ if and only if $u, v \in U_i, |u| = |v|$. Thus, there exist elements $u_1^i, \dots, u_n^i, \dots$ in U_i such that

$$U_i = \bigcup_j K_i(u_j^i).$$

By Zorn's Lemma we can assume $K_i(u_j^i), j \in \mathbb{N}$ to be well-ordered under some order $>_j$. If $u, v \in U_i$, then put

$$u > v \quad \text{iff} \quad \begin{cases} |u| > |v|, \\ \text{or,} \\ |u| = |v| \text{ but } u >_j v, \text{ where } u, v \in K_i(u_j^i) \text{ for some } u_j^i. \end{cases}$$

Observe that U_i is well-ordered under $>$ and enumerating its elements with respect to this order we get the set of indices I_i of elements from U_i . The order on U_i induces an order on $U = \bigcup_i U_i$ — if $u, v \in U_i$ then we compare them using the order defined on U_i and if $u \in U_i, v \in U_j$ then $u > v$ if and only if $i > j$.

Now we introduce the notion of abelian height for elements from $\mathbb{Z}[t]$. Recall that $\mathbb{Z}[t]$ is a countable direct sum

$$\mathbb{Z}[t] = \bigoplus_{i=0}^{\infty} \langle t^i \rangle$$

of copies of the infinite cyclic group \mathbb{Z} with the right lexicographic order. So, for every $\alpha \in \mathbb{Z}[t]$ there exists a natural number n such that α belongs to

$$\mathbb{Z}^n = \bigoplus_{i=0}^n \langle t^i \rangle.$$

Thus, we say that an *abelian height* $h_{ab}(\alpha) = n$ if $\alpha \in \mathbb{Z}^n - \mathbb{Z}^{n-1}$.

Compare the definition of abelian height with the definition of *height* given in [10]. Observe that if $w = u^\alpha \in R(\mathbb{Z}[t], X)$ then $h(w) = h(u)h_{ab}(\alpha)$.

2.2. Reduced forms for elements of $F^{\mathbb{Z}[t]}$

We start with the following convention, which somehow simplifies notations used throughout the text.

Conventions 2.1. From now on we call $\alpha \in \mathbb{Z}[t]$ *non-standard* if it does not belong to the “standard” part of $\mathbb{Z}[t]$ which is the smallest convex subgroup of $\mathbb{Z}[t]$. If α is non-standard then we write $\alpha \gg 0$.

An element $g \in G_{n+1} - G_n$ has the following representation as a reduced infinite word

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \dots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}, \tag{2.2}$$

where $n_1, n_2, \dots, n_l \in I_n$, $g_k \in G_n$, $k \in [1, l + 1]$, $[g_k, u_{n_k}] \neq \varepsilon$, $[g_{k+1}, u_{n_k}] \neq \varepsilon$, $k \in [1, l]$, $|\alpha_k| \gg 0$, $k \in [1, l]$. Observe that this representation is not unique because for each $k \in [1, l]$ it is possible that g_k has $u_{n_k}^{p_k}$ as a terminal segment and g_{k+1} has $u_{n_k}^{m_k}$ as an initial segment, so we can adjoin these finite exponents of u_{n_k} from the left and from the right to $u_{n_k}^{\alpha_k}$, to form a new infinite word representation of the same element g

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ h_2 \circ \dots \circ u_{n_l}^{\beta_l} \circ h_{l+1},$$

where $\beta_k = \alpha_k + p_k + m_k$, $g_1 = h_1 \circ u_{n_1}^{p_1}$, $g_{l+1} = u_{n_l}^{m_l} \circ h_{l+1}$, $g_k = u_{n_{k-1}}^{m_{k-1}} \circ h_k \circ u_{n_k}^{p_k}$, $k \in [2, l]$.

Remark 2.2. It is not hard to see that there are infinitely many representations of g in the form (2.2) — it is possible not only to perform the transformation described above, that is, to adjoin finite exponents of u_{n_k} , $k \in [1, l]$ from the left and from the right to $u_{n_k}^{\alpha_k}$, but also to perform inverse transformations, that is, to take finite exponents from each $u_{n_k}^{\alpha_k}$, $k \in [1, l]$ in order to obtain infinitely many tuples of interleaving elements f_1, \dots, f_{l+1} .

Lemma 2.3. *Let $g \in G_{n+1} - G_n$ have two representations*

$$g = g_1 \circ u_{s_1}^{\alpha_1} \circ g_2 \circ \dots \circ u_{s_l}^{\alpha_l} \circ g_{l+1},$$

where $s_j \in I_n$, $g_j \in G_n$, $[g_j, u_{s_j}] \neq \varepsilon$, $[g_{j+1}, u_{s_j}] \neq \varepsilon$, $|\alpha_j| \gg 0$, $j \in [1, l]$ and

$$g = h_1 \circ u_{r_1}^{\beta_1} \circ h_2 \circ \dots \circ u_{r_l}^{\beta_l} \circ h_{m+1},$$

where $t_k \in I_n$, $h_k \in G_n$, $[h_k, u_{r_k}] \neq \varepsilon$, $[h_{k+1}, u_{r_k}] \neq \varepsilon$, $|\beta_k| \gg 0$, $k \in [1, m]$. Then

- (1) $l = m$,
- (2) $u_{r_j} = u_{s_j}$, $j \in [1, l]$.

Proof. We have

$$(g_{l+1}^{-1} \circ u_{s_l}^{-\alpha_l} \circ \dots \circ u_{s_1}^{-\alpha_1} \circ g_1^{-1}) * (h_1 \circ u_{r_1}^{\beta_1} \circ h_2 \circ \dots \circ u_{r_l}^{\beta_l} \circ h_{m+1}) = \varepsilon.$$

By [10, Lemma 6.9] the equality above can hold only if $u_{r_1} = u_{s_1}$ and $g_1^{-1} * h_1 = u_{s_1}^{k_1}$. Thus, we have

$$(g_{l+1}^{-1} \circ u_{s_l}^{-\alpha_l} \circ \dots \circ g_2^{-1}) * u_{s_1}^{p_1} * (h_2 \circ \dots \circ u_{r_l}^{\beta_l} \circ h_{m+1}) = \varepsilon,$$

where p_1 is finite. Again, by [10, Lemma 6.9], $u_{r_2} = u_{s_2}$ and $g_2^{-1} * u_{s_1}^{p_1} * h_2 = u_{s_2}^{k_2}$ and we can cancel $(u_{s_2}^{-\alpha_2} \circ g_2^{-1}) * u_{s_1}^{p_1} * (h_2 \circ u_{r_2}^{\beta_2})$ into a finite exponent $u_{s_2}^{p_2}$. The required result now follows by induction. \square

It follows from Lemma 2.3 that in any representation of $g \in G_{n+1} - G_n$ as an infinite word, the number of non-standard exponents $u_{n_k}^{\alpha_k}$ in g (the number of syllables in some sense) is the same, which is equal to l . This observation makes it possible to introduce a natural characterization of any representation of g as an l -tuple $\{|\gamma_1|, |\gamma_2|, \dots, |\gamma_l|\}$, where $|\gamma_k| \gg 0$ and

$$g = f_1 \circ u_{n_1}^{\gamma_1} \circ f_2 \circ \dots \circ u_{n_l}^{\gamma_l} \circ f_{l+1}. \tag{2.3}$$

We call such a representation of g , U_n -reduced if the ordered l -tuple $\{|\gamma_1|, |\gamma_2|, \dots, |\gamma_l|\}$ is maximal with respect to the right lexicographic order among all possible representations of g .

In [10] we introduced unique reduced forms for elements of $F^{\mathbb{Z}[t]}$. Now, we redefine the procedure for obtaining them and show that in fact unique reduced forms defined in the proof of [10, Theorem 6.15] coincide with U_n -reduced representations (in the future we will use the term U_n -form instead of U_n -reduced representation).

Suppose $g \in G_{n+1} - G_n$ and

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \dots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}.$$

From [10, Lemma 6.9] it follows that for g_1 and g_2 there exist $p_1, m_1 \in \mathbb{N}$ such that $g_1 = h_1 \circ u_{n_1}^{p_1}$, $g_2 \circ u_{n_2}^{\alpha_2} = u_{n_1}^{m_1} \circ g'_2 \circ u_{n_2}^{\gamma_2}$ and h_1 does not have $u_{n_1}^{\pm 1}$ as a terminal segment, $g'_2 \circ u_{n_2}^{\gamma_2}$ does not have $u_{n_1}^{\pm 1}$ as an initial segment. Now we present g as

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ g'_2 \circ u_{n_2}^{\gamma_2} \circ \dots \circ u_{n_l}^{\alpha_l} \circ g_{l+1},$$

where $\beta_k = \alpha_k + p_k + m_k$, $g_1 = h_1 \circ u_{n_1}^{p_1}$, $g_2 \circ u_{n_2}^{\alpha_2} = u_{n_1}^{m_1} \circ g'_2 \circ u_{n_2}^{\gamma_2}$. Next we take the subword of g

$$g' = g'_2 \circ u_{n_2}^{\gamma_2} \circ g_3 \circ \dots \circ u_{n_l}^{\alpha_l} \circ g_{l+1}$$

and perform exactly the same procedure of maximizing γ_2 . The whole construction follows by induction on l . After a finite number of steps we get

$$g = h_1 \circ u_{n_1}^{\beta_1} \circ h_2 \circ \dots \circ u_{n_l}^{\beta_l} \circ h_{l+1}. \tag{2.4}$$

Lemma 2.4.

- (1) (2.4) is unique,
- (2) (2.4) is a U_n -reduced form for g .

Proof. (1) follows immediately from the construction.

(2) Suppose (2.4) is not a U_i -reduced form for g , that is, there exists an l -tuple $\{|\gamma_1|, |\gamma_2|, \dots, |\gamma_l|\}$ such that $\{|\gamma_1|, |\gamma_2|, \dots, |\gamma_l|\} > \{|\beta_1|, |\beta_2|, \dots, |\beta_l|\}$ and

$$g = f_1 \circ u_{n_1}^{\gamma_1} \circ f_2 \circ \dots \circ u_{n_l}^{\gamma_l} \circ f_{l+1}.$$

From the inequality above it follows that there exists $k_0 \in [1, l]$ such that $\gamma_k = \beta_k$, $k \in [1, k_0 - 1]$ and $\gamma_{k_0} > \beta_{k_0}$.

At first we show that $h_k = f_k$, $k \in [1, k_0 - 1]$. Assume that $k_0 > 1$.

Observe that $|h_1| \leq |f_1|$ otherwise $h_1 = f_1 \circ u_{n_1}^m$, $|m| > 0$, $m \in \mathbb{Z}$ and β_1 is not maximal — contradiction (here we use [10, Lemma 6.9] from which it follows that $f_1^{-1} * h_1 \in \langle u_{n_1} \rangle$). If $|h_1| < |f_1|$ then since $\beta_1 = \gamma_1$ it follows that $h_2 \circ u_{n_2}^{\beta_2}$ contains u_{n_1} as an initial segment — contradiction with the choice of β_1 . Thus $|h_1| = |f_1|$ and $h_1 = f_1$. In the same way, using induction one can prove that $h_k = f_k$, $k \in [2, k_0 - 1]$.

The simple fact proved above shows that we can assume without loss of generality that $k_0 = 1$, that is, $\gamma_1 > \beta_1 > 0$. Again, $|h_1| \leq |f_1|$ since otherwise we get a contradiction. As above if $|h_1| < |f_1|$ then since $\beta_1 < \gamma_1$ it follows that $h_2 \circ u_{n_2}^{\beta_2}$ contains at least u_{n_1} as an initial segment — contradiction with the choice of β_1 .

Thus we proved that $\{|\beta_1|, |\beta_2|, \dots, |\beta_l|\}$ is the maximal l -tuple with respect to the right lexicographical order. So the representation (2.4) is U_n -reduced for g . \square

Let $g \in G_{n+1} - G_n$, so it has a representation as an infinite word

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \dots \circ u_{n_l}^{\alpha_l} \circ g_{l+1},$$

where $n_1, n_2, \dots, n_l \in I_n$, $g_k \in G_n$, $[g_k, u_{n_k}] \neq \varepsilon$, $[g_{k+1}, u_{n_k}] \neq \varepsilon$, $|\alpha_k| \gg 0$. Now we fix some u from the list $u_{n_1}, u_{n_2}, \dots, u_{n_l}$ of elements from U_n taken to non-standard powers in the representation of g (it follows from Lemma 2.3 that this list does not depend on particular representation of g as a reduced infinite word). Consider now a representation of g in which we “mark” only non-standard exponents of u , that is,

$$g = h_1 \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}, \tag{2.5}$$

where $\beta_j = \alpha_{m_j}$, $m_j \in [1, l]$, $j \in [1, p]$, $h_1 = g_1 \circ u_{n_1}^{\alpha_1} \circ \dots \circ g_{m_1}$, $h_{p+1} = g_{m_p+1} \circ \dots \circ g_{l+1}$, $h_k = g_{m_k+1} \circ \dots \circ g_{m_k+1}$, $k \in [2, p]$. Observe that all h_k , in general, do not belong to G_n any more. We call (2.5) the *the u -representation* or *the u -form* for g .

One can prove a statement analogous to Lemma 2.3 for u -forms of g and this means that we can associate with any such form a p -tuple $\{|\beta_1|, |\beta_2|, \dots, |\beta_p|\}$ of elements from $\mathbb{Z}[t]$, which are infinite exponents from (2.5).

We call a u -form of g , *u -reduced* if the ordered p -tuple $\{|\beta_1|, |\beta_2|, \dots, |\beta_p|\}$ is maximal with respect to the right lexicographic order among all possible u -forms for g .

Suppose

$$h_1 \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1} \tag{2.6}$$

is a u -form for g and g is cyclically reduced. Then, obviously

$$(h_1 \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}) \circ (h_1 \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}) \tag{2.7}$$

is a u -form of g^2 . So, we call (2.6) *cyclically u -reduced* if (2.7) is u -reduced.

Lemma 2.5. *Let*

$$h_1 \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}$$

be a u -reduced form of $g \in G_{n+1} - G_n$, $u \in U_i$. Then there exists a cyclic permutation of g such that its u -reduced form is cyclically u -reduced.

Proof. The proof is based on the following observation.

Claim 2.6. *Let $|w| < |u|$. If $w \circ u^\beta$ has u as an initial segment in $u^\alpha \circ w \circ u^\beta$, $\alpha, \beta \in \mathbb{Z}$, $\alpha, \beta > 0$ then $u^\alpha \circ w$ cannot have u as a terminal segment.*

Suppose on the contrary that $u = w \circ u_1$, $u = u_1 \circ u_2$ and $u = u_3 \circ w$, $u = u_4 \circ u_3$ at the same time. From these equalities we have $|w| = |u_2|$, $|u_1| = |u_3|$, $|u_4| = |w|$. Since $u = w \circ u_1$, $u = u_4 \circ u_3$ and $|u_4| = |w|$ it follows that $w = u_4$, $u_1 = u_3$. Thus $w = u_2$ and $u = u_1 \circ u_2 = u_2 \circ u_1$, but this is possible only if $u_1 = a^\delta$ and $u_2 = a^\gamma$, and hence $u = a^{\delta+\gamma}$, so the centralizer of u in G_n is not cyclic — contradiction with the choice of U_n .

Observe also that if $\alpha > 0$, $\beta < 0$ then neither $w \circ u^\beta$ has u as an initial segment nor $u^\alpha \circ w$ has u^{-1} as a terminal segment. Indeed, if for example $w \circ u^\beta$ has u as an initial segment then $u = w \circ u_1$, $u^{-1} = u_1 \circ u_2$. Thus, $u = w \circ u_1 = u_2^{-1} \circ u_1^{-1}$ and $u_1 = u_1^{-1} = \varepsilon$, $u = w$ — contradiction.

Now we complete the proof of the lemma. Without loss of generality we can assume $h_1 = \varepsilon$ and that h_{p+1} does not have u as a terminal segment (using cyclic permutation we can always obtain these properties). We have two cases

(1) $|h_{p+1}| \geq |u|$

Since $u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}$ is u -reduced it follows that $u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1} \circ u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}$ is u -reduced because h_{p+1} does not have u as an initial or terminal segment.

(2) $|h_{p+1}| < |u|$

(a) If $u^{\beta_p} \circ h_{p+1}$ has u as a terminal segment then by Claim 2.6, $h_{p+1} \circ u^{\beta_1}$ does not have u as an initial segment. Since $u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}$ is u -reduced then the ordered $2p$ -tuple $\{|\beta_1|, |\beta_2|, \dots, |\beta_p|, |\beta_1|, |\beta_2|, \dots, |\beta_p|\}$ is maximal with respect to the right lexicographic order among all possible u -forms of g^2 . So, $u^{\beta_1} \circ h_2 \circ \dots \circ u^{\beta_p} \circ h_{p+1}$ is cyclically u -reduced.

(b) If $h_{p+1} \circ u^{\beta_1}$ has u as an initial segment then by Claim 2.6, $u^{\beta_p} \circ h_{p+1}$ does not have u as a terminal segment. Thus, take a cyclic permutation $g' = h_2 \circ u^{\beta_2} \dots \circ u^{\beta_p} \circ h_{p+1} \circ u^{\beta_1}$ of g so that $h_2 \circ u^{\beta_2} \dots \circ u^{\beta_p+1} \circ b \circ u^{\beta_1-1}$ is a u -reduced form of g' , where $u = h_{p+1} \circ a = a \circ b$. But then $(h_2 \circ u^{\beta_2} \dots \circ u^{\beta_p+1} \circ b \circ u^{\beta_1-1}) \circ (h_2 \circ u^{\beta_2} \dots \circ u^{\beta_p+1} \circ b \circ u^{\beta_1-1})$ is u -reduced. □

2.3. Standard decomposition of elements of $F^{\mathbb{Z}[t]}$

In this section, from infinite word representation of elements from $F^{\mathbb{Z}[t]}$ we obtain their representation by finite words in some alphabet.

Let $g \in F^{\mathbb{Z}[t]}$. Then $g \in G_{n+1} - G_n$ for some $n \in \mathbb{N}$ and g has a U_n -reduced form

$$g = g_1 \circ u_{n_1}^{\alpha_1} \circ g_2 \circ \cdots \circ u_{n_l}^{\alpha_l} \circ g_{l+1},$$

where $u_{n_1}, u_{n_2}, \dots, u_{n_l} \in U_n$, $g_k \in G_n$, $k \in [1, l+1]$, $[g_k, u_{n_k}] \neq \varepsilon$, $[g_{k+1}, u_{n_k}] \neq \varepsilon$, $|\alpha_k| \gg 0$, $k \in [1, l]$. Now, by the induction on n we can assume existence of a U_{n-1} -reduced form for each g_i , $i \in [1, l]$

$$g_i = g(i)_1 \circ u_{m_1}^{\beta_{m_1}} \circ g(i)_2 \circ \cdots \circ u_{m_s}^{\beta_{m_s}} \circ g(i)_s,$$

where $u_{m_1}, \dots, u_{m_s} \in U_{n-1}$, $|\beta_{m_k}| \gg 0$, $k \in [1, s]$, $g(i)_k \in G_{n-1}$, $k \in [1, s+1]$. One can get down to the free group F with such a decomposition of g , where subwords between non-standard powers of elements from U_i are represented step by step as U_{i-1} -forms, $i \in [1, n]$. Thus, from this decomposition one can form the following series for g :

$$\begin{aligned} F < H_{0,1} < H_{0,2} < \cdots < H_{0,k(0)} < H_{1,1} < \cdots < H_{1,k(1)} \\ < \cdots < H_{n-1,k(n-1)} < H_{n,1} < \cdots < H_{n,k(n)}, \end{aligned} \quad (2.8)$$

where $H_{j,1}, \dots, H_{j,k(j)}$ are subgroups of G_{j+1} which do not belong to G_j and $H_{j,i}$ is obtained from $H_{j,i-1}$ by a centralizer extension of a single element $u_{j,i-1} \in H_{j,i-1} < G_j$. Element g belongs to $H_{n,k(n)}$ and does not belong to the previous terms. Series (2.8) is called *the extension series* for g .

Using the extension series above we can decompose g in the following way: $g \in H_{n,k(n)}$ has a $u_{n,k(n)}$ -reduced form

$$g = h_1 \circ u_{n,k(n)}^{\beta_1} \circ h_2 \circ \cdots \circ u_{n,k(n)}^{\beta_l} \circ h_{l+1},$$

where all $h_j, j \in [1, l+1]$ in their turn are $u_{n,k(n)-1}$ -reduced forms representing elements from $H_{n,k(n)-1}$. This gives one a decomposition of g related to its extension series. We call this decomposition the *standard decomposition* or *standard representation* of g .

Observe that for any $g \in F^{\mathbb{Z}[t]}$, its standard decomposition can be viewed as a finite word in the alphabet

$$B = X^\pm \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t] - \mathbb{Z}\}.$$

We denote this product by $\pi(g)$ so we have

$$\pi(g) = \pi(h_1) u_{n,k(n)}^{\beta_1} \pi(h_2) \cdots u_{n,k(n)}^{\beta_l} \pi(h_{l+1}),$$

where $\pi(h_i)$ is a finite product in the alphabet B corresponding to h_i , and from now on, by the standard decomposition of an element g we understand not the representation of g as a reduced infinite word but the finite word $\pi(g)$.

Let $U(g)$ be a finite subset of U such that $u \in U(g)$ only if $\pi(g)$ contains a letter $b_i \in B$ such that $b_i = u^\alpha$, $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$. Observe that $U(g)$ is ordered with respect to the order induced from U , that is, we have

$$U(g) = \{u_1, \dots, u_m\},$$

where $u_i < u_j$ if $i < j$ and $u_m = u_{n,k(n)}$. By $\max\{U(g)\}$ we denote the maximal element of $U(g)$.

If $u \in U(g)$ then by $deg_u(g)$ we denote the maximal degree of infinite exponents of u which appear in $\pi(g)$.

It is easy to see that in general $\pi(g_1 \circ g_2) \neq \pi(g_1)\pi(g_2)$ and $\pi(g \circ g) = \pi(g)\pi(g)$ if and only if the u -reduced form of g is cyclically u -reduced, where $u = \max\{U(g)\}$.

From the definition of Lyndon's set and the results of [10] it follows that if $R \subset G_n$ is a Lyndon's set then a set R' obtained from R by cyclic decompositions of its elements is also a Lyndon's set. Thus, by Lemma 2.5 we can assume a w -reduced form of any $u \in U_n$ to be cyclically w -reduced, where $w = \max\{U(u)\}$. Hence, we can assume

$$\pi(u \circ u) = \pi(u)\pi(u)$$

for any $u \in U$.

3. Graphs Labeled by Infinite $\mathbb{Z}[t]$ -Words

In this section we introduce the notion of $(\mathbb{Z}[t], X)$ -graphs and describe their basic properties.

3.1. Labeled graphs

Using the notation introduced in the previous section we adjust basic notions from [4] to $F^{\mathbb{Z}[t]}$.

Definition 3.1. By a $(\mathbb{Z}[t], X)$ -labeled directed graph ($(\mathbb{Z}[t], X)$ -graph) Γ we mean the following.

- (1) Γ is a combinatorial graph where every edge has a direction and is labeled either by a letter from X or by an infinite exponent $u^\alpha \in F^{\mathbb{Z}[t]}$, $u \in U$, $\alpha \in \mathbb{Z}[t]$, $\alpha > 0$, denoted $\mu(e)$;
- (2) for each edge e of Γ we denote the origin of e by $o(e)$ and the terminus of e by $t(e)$.

For each edge e of $(\mathbb{Z}[t], X)$ -graph we introduce a formal inverse e^{-1} of e with label $\mu(e)^{-1}$ and the endpoints defined as $o(e^{-1}) = t(e)$, $t(e^{-1}) = o(e)$, that is, the direction of e^{-1} is reversed with respect to the direction of e . For the new edge e^{-1} we set $(e^{-1})^{-1} = e$. The new graph, endowed with this additional structure we denote by $\widehat{\Gamma}$. In fact, usually we will abuse notation by disregarding the difference between Γ and $\widehat{\Gamma}$.

Now we have a partition $E(\widehat{\Gamma}) = E(\Gamma) \cup \overline{E(\Gamma)}$ and we say that edges of Γ are *positively oriented* in $\widehat{\Gamma}$, while their formal inverses e^{-1} are *negatively oriented* in $\widehat{\Gamma}$.

Definition 3.2. A path p in Γ is a sequence of edges $p = e_1 \cdots e_k$, where each e_i is an edge of $\widehat{\Gamma}$ and the origin of each e_i is the terminus of e_{i-1} .

Observe that $\mu(p) = \mu(e_1) \cdots \mu(e_k)$ is a word in the alphabet $X^\pm \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t]\}$ and we denote by $\overline{\mu(p)}$ the reduced infinite word $\mu(e_1) * \cdots * \mu(e_k)$.

We will be using two different notions of the length of a path $p = e_1 \cdots e_k$ in Γ

- (1) *combinatorial length* $|p|$ set equal to k , and
- (2) *word length* $wl(p) = \sum_{i=1}^k l(\mu(e_i))$.

In fact, from these two definitions above two possible meanings of an irreducible path arise: an irreducible path in combinatorial sense and an irreducible path in the sense that its label viewed as an infinite word in $F^{\mathbb{Z}[t]}$ is reduced. Here are formal definitions.

Definition 3.3. A path $p = e_1 \cdots e_k$ in a $(\mathbb{Z}[t], X)$ -graph Γ is called reduced if $e_i \neq e_{i+1}^{-1}$ for all $i \in [1, k - 1]$.

Definition 3.4. A path $p = e_1 \cdots e_k$ in a $(\mathbb{Z}[t], X)$ -graph Γ is called *label reduced* if

- (1) p is reduced;
- (2) $e_{k_1} \cdots e_{k_2}$, $k_1 \leq k_2$ is a subpath of p such that $\mu(e_i) = u^{\alpha_i}$, $u \in U$, $\alpha_i \in \mathbb{Z}[t]$, $i \in [k_1, k_2]$ and $\mu(e_{k_1-1}) \neq u^{\beta_1}$, $\mu(e_{k_2+1}) \neq u^{\beta_2}$ for any $\beta_1, \beta_2 \in \mathbb{Z}[t]$, provided $k_1 - 1, k_2 + 1 \in [1, k]$, then $\alpha = \alpha_{k_1} + \cdots + \alpha_{k_2} \neq 0$ and $\mu(e_{k_1-1}) * u^\alpha = \mu(e_{k_1-1}) \circ u^\alpha$, $u^\alpha * \mu(e_{k_2+1}) = u^\alpha \circ \mu(e_{k_2+1})$.

3.2. Free foldings

Here we define free (partial) foldings and partially folded $(\mathbb{Z}[t], X)$ -graphs. Observe that the definition of a partial folding below is exactly the same as the corresponding definition of a folding in free groups (see [4]).

Definition 3.5. Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph and let v be a vertex of Γ . The valence of v denoted $val(v)$ is the number of all edges in Γ which have v as an origin or a terminus.

Let Γ be a $(\mathbb{Z}[t], X)$ -graph. Suppose v_0 is a vertex of Γ and f_1, f_2 are two distinct edges of $\widehat{\Gamma}$ such that $o(f_1) = o(f_2) = v_0$, $\mu(f_1) = \mu(f_2) = x \in X^\pm$ or $\mu(f_1) = \mu(f_2) = u^\alpha$, $u \in U$, $\alpha \in \mathbb{Z}[t]$. Let h_i be the positive edge of Γ corresponding to f_i (that is, $h_i = f_i$ if f_i is positive and $h_i = f_i^{-1}$ if f_i is negative).

Let Δ be a $(\mathbb{Z}[t], X)$ -graph with the following sets of vertices and edges.

$$V(\Delta) = (V(\Gamma) - \{t(f_1), t(f_2)\}) \cup \{v\}, \quad E(\Delta) = (E(\Gamma) - \{h_1, h_2\}) \cup \{h\}.$$

The endpoints and arrows for the edges of Δ are defined in the following way. Let $e \in E(\Delta)$, $e \neq h$ then

- (1) we put $o_\Delta(e) = o_\Gamma(e)$ if $o_\Gamma(e) \neq t(f_i)$ and $o_\Delta(e) = v$ if $o_\Gamma(e) = t(f_i)$ for some i ,
- (2) we put $t_\Delta(e) = t_\Gamma(e)$ if $t_\Gamma(e) \neq t(f_i)$ and $t_\Delta(e) = v$ if $t_\Gamma(e) = t(f_i)$ for some i .

For the edge h we put $o_\Delta(h) = v_0$, $t_\Delta(h) = v$ if $h_1 = f_1$, $h_2 = f_2$ and $o_\Delta(h) = v$, $t_\Delta(h) = v_0$ otherwise.

We define labels on the edges of Δ as follows: $\mu_\Delta(e) = \mu_\Gamma(e)$ if $e \neq h$ and $\mu_\Delta(h) = \mu_\Gamma(h_1) = \mu_\Gamma(h_2)$.

In other words we obtain Δ by identification of two edges f_1 and f_2 in Γ . In this situation we say that Δ is obtained from Γ by a *free (partial) folding* (or by *freely (partially) folding the edges f_1 and f_2*).

There can be introduced a notion of a *morphism* between two $(\mathbb{Z}[t], X)$ -graphs. That is, if Γ_1, Γ_2 are $(\mathbb{Z}[t], X)$ -graphs then a map $\theta: \Gamma_1 \rightarrow \Gamma_2$ is called a *morphism* of $(\mathbb{Z}[t], X)$ -graphs, if θ sends vertices to vertices, directed edges to directed edges, preserves labels of directed edges, and has the property that $o(\theta(e)) = \theta(o(e))$, $t(\theta(e)) = \theta(t(e))$ for any edge e of Γ_1 .

If ϕ is a partial folding defined above then it is easy to see that ϕ is a morphism between Γ and Δ .

Lemma 3.6. *Let Γ_1 be a $(\mathbb{Z}[t], X)$ -graph obtained by a free folding from a graph Γ . Let v be a vertex of Γ and v_1 be the corresponding vertex of Γ_1 . Then the following hold.*

- (1) *If Γ is connected then Γ_1 is connected.*
- (2) *Let p be the path from v to v in Γ with label w . Then the edgewise image of p in Γ_1 is a path from v_1 to v_1 with label w .*
- (3) *If Γ is a finite $(\mathbb{Z}[t], X)$ -graph, then the number of edges in Γ_1 is one less than the number of edges in Γ , that is, any free folding decreases the number of edges in Γ .*

Proof. Follows directly from the definition of a free folding. □

Definition 3.7. $(\mathbb{Z}[t], X)$ -graph Γ is called *partially folded* if there exist no two edges e_1 and e_2 in Γ with $\mu(e_1) = \mu(e_2)$ such that $o(e_1) = o(e_2)$ or $t(e_1) = t(e_2)$.

Obviously, Γ is a partially folded $(\mathbb{Z}[t], X)$ -graph if and only if one cannot perform any free folding in Γ . Moreover the following proposition is true.

Proposition 3.8. *Let Γ be a $(\mathbb{Z}[t], X)$ -graph, which has only a finite number of edges. Then there exists a partially folded $(\mathbb{Z}[t], X)$ -graph Δ , which can be obtained from Γ by a finite number of free foldings.*

Proof. Since Γ has a finite number of edges by Lemma 3.6 any $(\mathbb{Z}[t], X)$ -graph Γ_1 obtained from Γ by a free folding has fewer edges. This provides one with an inductive argument based on the number of edges in Γ . □

3.3. u -components

In the present subsection we concentrate on some particular subgraphs of $(\mathbb{Z}[t], X)$ -graphs which consist of edges labeled by exponents of elements from U . These subgraphs are very important in all our further investigations.

Let $u \in U$ be fixed.

Definition 3.9. Let Γ be a $(\mathbb{Z}[t], X)$ -graph. Vertices $v_1, v_2 \in V(\Gamma)$ are called u -equivalent ($v_1 \sim_u v_2$) if there exists a path $p = e_1 \cdots e_k$ in Γ such that $o(e_1) = v_1$, $t(e_k) = v_2$ and $\mu(e_i) = u_i^\alpha$, $\alpha_i \in \mathbb{Z}[t]$, $i \in [1, k]$.

\sim_u is clearly an equivalence relation on vertices of Γ , so if Γ is finite then all its vertices can be divided into a finite number of pairwise disjoint equivalence classes. Suppose $v \in V(\Gamma)$ is fixed. One can take the subgraph of Γ spanned by vertices u -equivalent to v and remove from it all edges labeled by anything except for u^α , $\alpha \in \mathbb{Z}[t]$. The resulting subgraph of Γ we denote by $Comp_u(v)$ and call the u -component of v . In other words, the u -component of a vertex v is the subgraph of Γ all edges of which are labeled by exponents of u .

Definition 3.10. Let Γ be a $(\mathbb{Z}[t], X)$ -graph and $v \in V(\Gamma)$, $v_0 \in V(Comp_u(v))$. We define a set $H_u(v_0)$ associated with v_0 as

$$H_u(v_0) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } Comp_u(v) \text{ from } v_0 \text{ to } v_0\}.$$

Observe that even when p is a reduced path in $Comp_u(v)$ its label $\mu(p)$ may be the empty infinite word.

Lemma 3.11. *Let Γ be a $(\mathbb{Z}[t], X)$ -graph and $v \in V(\Gamma)$, $v_0 \in V(Comp_u(v))$. Then*

- (1) $H_u(v_0)$ is isomorphic to a subgroup of $\mathbb{Z}[t]$ and moreover, if $Comp_u(v)$ is a finite graph then $H_u(v_0)$ is finitely generated,
- (2) if $v_1 \in V(Comp_u(v))$ then $H_u(v_0) \simeq H_u(v_1)$.

Proof. (1) Observe that if p is a cycle in $Comp_u(v)$ at v_0 then $\overline{\mu(p)} = u^\alpha$, $\alpha \in \mathbb{Z}[t]$. The concatenation $p_1 p_2$ of two cycles in $Comp_u(v)$ at v_0 is again a cycle in $Comp_u(v)$ at v_0 which may or may not be reduced. Let p be the reduced cycle obtained from $p_1 p_2$ by making all possible path reductions. Then $\overline{\mu(p)} = \overline{\mu(p_1)} * \overline{\mu(p_2)} = \overline{\mu(p_2)} * \overline{\mu(p_1)} \in H_u(v_0)$ and $H_u(v_0)$ is closed under multiplication which is commutative.

Finally, since the inverse path $(p_1)^{-1}$ of p_1 is reduced and is labeled by $\mu(p_1)^{-1}$, it follows that $H_u(v_0)$ is closed under taking inverses. Also, since ε is a label of an empty path which is reduced then clearly $\varepsilon \in H_u(v_0)$.

Thus, $H_u(v_0)$ is an abelian group.

One can construct a map $\theta : H_u(v_0) \rightarrow \mathbb{Z}[t]$, where $\overline{\mu(p)} = u^\alpha \xrightarrow{\theta} \alpha \in \mathbb{Z}[t]$. Obviously, θ is an isomorphism of $H_u(v_0)$ and a subgroup of $\mathbb{Z}[t]$. Moreover, if $Comp_u(v)$ is a finite graph then there exists a natural number n such that $n \geq h_{ab}(\alpha)$, where $\mu(e) = u^\alpha$ and e ranges through all edges of $Comp_u(v)$. Since for any reduced cycle $p = e_1 \cdots e_k$ in $Comp_u(v)$ such that $\mu(e_i) = u^{\alpha_i}$, $i \in [1, k]$ and

$\overline{\mu(p)} = u^{\alpha_1 + \dots + \alpha_k} = u^\alpha$, one has $h_{ab}(\alpha) \leq \max_{i=1}^k \{h_{ab}(\alpha_i)\}$ and it follows that $H_u(v_0)$ is a subgroup of \mathbb{Z}^n which is a finite rank subgroup of $\mathbb{Z}[t]$. Thus $H_u(v_0)$ is a free abelian group of rank not greater than n .

(2) Since $v_0 \sim_u v_1$, there exists a path $p = f_1 \cdots f_k$ in $Comp_u(v)$ such that $o(p) = v_0$, $t(p) = v_1$ and $\overline{\mu(p)} = u^\gamma$, $\gamma \in \mathbb{Z}[t]$. The existence of p provides one with correspondence between cycles at v_0 and v_1 — if $p_0 = e_1 \cdots e_m$ is a reduced cycle at v_1 then $p_1 = f_1 \cdots f_k e_1 \cdots e_m (f_k)^{-1} \cdots (f_1)^{-1}$ is a cycle at v_0 and one can obtain a reduced cycle p_2 at v_0 by making all possible path reductions in p_1 . Observe that $\overline{\mu(p_2)} = \overline{\mu(p_1)} = \overline{\mu(p)} * \overline{\mu(p_0)} * (\overline{\mu(p)})^{-1} = \overline{\mu(p_0)}$. Thus $H_u(v_1) \leq H_u(v_0)$. In the same way for any cycle at v_0 one can construct a corresponding cycle at v_1 with the same label. So $H_u(v_1) \geq H_u(v_0)$ and hence $H_u(v_1) \simeq H_u(v_0)$. \square

It follows from Lemma 3.11 that one can associate a free abelian group of finite rank with any finite u -component in a $(\mathbb{Z}[t], X)$ -graph Γ .

3.4. u -folded u -components

In Sec. 3.2 we introduced a notion of free foldings, but it turns out that for our further investigations it is not enough to have only partially folded graphs. In the present subsection we introduce the notion of U -foldings which are operations on partially folded $(\mathbb{Z}[t], X)$ -graphs.

Let Γ be a $(\mathbb{Z}[t], X)$ -graph and let $u \in U$ be fixed throughout this subsection. Let $v \in V(\Gamma)$, $v_0 \in V(Comp_u(v))$.

Free foldings defined in the previous subsection are simple operations on edges which have the same labels, but they do not cover the case when we have two edges $f_1, f_2 \in E(Comp_u(v))$ such that $o(f_1) = o(f_2) = v_0$, $\mu(f_1) = u^\alpha$, $\mu(f_2) = u^\beta$, $u \in U$, $\alpha, \beta \in \mathbb{Z}[t]$, $\alpha \neq \beta$. Since $\alpha \neq \beta$, no free folding can be applied.

Definition 3.12 (u-folding). Let $f_1, f_2 \in E(Comp_u(v))$ be such that $o(f_1) = o(f_2) = v_0$ such that $\mu(f_1) = u^\alpha$, $\mu(f_2) = u^\beta$, $\alpha, \beta \in \mathbb{Z}[t]$ and suppose $|\alpha| \geq |\beta|$. Without loss of generality we can assume both edges to be positively oriented, that is, $\alpha, \beta > 0$ (otherwise we can consider f_i^{-1} , $i = 1, 2$ instead of f_i).

Let Γ_1 be a $(\mathbb{Z}[t], X)$ -graph defined as follows.

$$V(\Gamma_1) = V(\Gamma) \cup \{v_1\}, \quad E(\Gamma_1) = (E(\Gamma) - \{f_1\}) \cup \{e_1, e_2\}.$$

We think of Γ_1 as a new $(\mathbb{Z}[t], X)$ -graph obtained from Γ by dividing the edge f_1 into two edges e_1 and e_2 . The endpoints and arrows for the edges of Γ_1 are defined in the following way. Let $e \in E(\Gamma_1)$, $e \neq e_1, e_2$ then we put $o_{\Gamma_1}(e) = o_\Gamma(e)$ and $t_{\Gamma_1}(e) = t_\Gamma(e)$

For the edges e_1, e_2 we put $o_{\Gamma_1}(e_1) = o_\Gamma(f_1)$, $t_{\Gamma_1}(e_1) = v_1$, $o_{\Gamma_1}(e_2) = v_1$, $t_{\Gamma_1}(e_2) = t_\Gamma(f_1)$.

Finally, $\mu_{\Gamma_1}(e) = \mu_\Gamma(e)$ if $e \neq e_1, e_2$ and $\mu_{\Gamma_1}(e_1) = \mu_\Gamma(f_2)$, $\mu_{\Gamma_1}(e_2) = u^{\alpha-\beta}$.

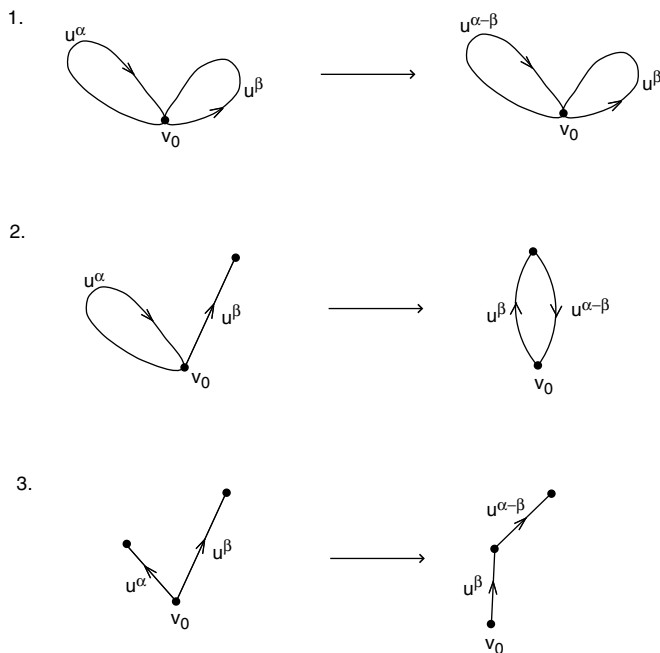


Fig. 1. Possible u -foldings.

Thus, in Γ_1 we have a pair of edges e_1, f_2 with origin v_0 and the same label u^β , so we can apply a free folding ψ to Γ_1 . After the identification of e_1 and f_2 the resulting $(\mathbb{Z}[t], X)$ -graph we denote by Δ .

In this situation we say that Δ is obtained from Γ by a u -folding (or by u -folding the edges f_1 and f_2) (see Fig. 1).

Observe that if $\alpha = \beta$ then the u -folding defined above is just a free folding.

From now on by U -foldings we denote the set of all u -foldings, where $u \in U$.

Unlike free foldings, u -foldings do not define morphisms of $(\mathbb{Z}[t], X)$ -graphs because they involve the operation of division of an edge. Hence, we introduce a notion of *generalized morphism* between two $(\mathbb{Z}[t], X)$ -graphs. That is, if Γ_1, Γ_2 are $(\mathbb{Z}[t], X)$ -graphs then a map $\theta : \Gamma_1 \rightarrow \Gamma_2$ is called a *generalized morphism* of $(\mathbb{Z}[t], X)$ -graphs, if θ sends vertices to vertices, directed edges to reduced paths and has the property that $o(\theta(p)) = \theta(o(p)), t(\theta(p)) = \theta(t(p)), \overline{\mu(\theta(p))} = \overline{\mu(p)}$ for any reduced path p of Γ_1 .

Let ϕ be a u -folding defined above which is applied to the pair of edges $\{f_1, f_2\}$ in $E(Comp_u(v))$. By definition, ϕ involves as a final stage a free folding ψ . Then we have $\phi(f_1) = \psi(e_1)\psi(e_2)$, $\phi(e) = \psi(e)$, $e \in E(\Gamma)$, $e \neq f_1$ and $\phi(w) = \psi(w)$, $w \in V(\Gamma)$. Observe that it follows from the definition of ϕ that $o(\phi(e)) = o(\psi(e)) = \psi(o(e)) = \phi(o(e))$, $t(\phi(e)) = t(\psi(e)) = \psi(t(e)) = \phi(t(e))$, $\mu(\phi(e)) = \mu(e)$ for any $e \in E(\Gamma)$, $e \neq f_1$ and $o(\phi(f_1)) = o(\psi(e_1)) = \psi(o(e_1)) = \phi(o(f_1))$, $t(\phi(f_1)) = t(\psi(e_2)) = \psi(t(e_2)) = \phi(t(f_1))$, $\overline{\mu(\phi(f_1))} = \overline{\mu(f_1)}$. Hence, it follows

that $o(\phi(p)) = \phi(o(p))$, $t(\phi(p)) = \phi(t(p))$, $\overline{\mu(\phi(p))} = \overline{\mu(p)}$ for any reduced path p in Γ . Observe that $\phi(p)$ may not be reduced.

Thus, we verified that u -folding is a generalized morphism of $(\mathbb{Z}[t], X)$ -graphs. The following result is analogous to Lemma 3.6 about free foldings.

Lemma 3.13. *Let Γ_1 be a $(\mathbb{Z}[t], X)$ -graph obtained by a u -folding from a $(\mathbb{Z}[t], X)$ -graph Γ . Let v be a vertex of Γ and v_1 be the corresponding vertex of Γ_1 . Then the following hold.*

- (1) *If Γ is connected then Γ_1 is connected.*
- (2) *Let p be the path from v to v in Γ such that $\overline{\mu(p)} = w$. Then the image of p in Γ_1 is a path p_1 from v_1 to v_1 such that $\overline{\mu(p_1)} = w$.*
- (3) *If Γ is a finite $(\mathbb{Z}[t], X)$ -graph, then $|V(\Gamma_1)| \leq |V(\Gamma)|$.*

Proof. (1) and (3) follow directly from the definition and (2) follows from the fact that u -foldings are generalized morphisms. □

It is easy to see the difference between Lemma 3.6 and Lemma 3.13 above. Unlike free foldings, u -foldings do not preserve labels of paths just because of the division of edges involved, but any path in Γ and its image in Γ_1 have the same labels viewed as reduced infinite words.

The following important result follows directly from Lemma 4.3 which will be proved in Sec. 4.2 in more general context.

Lemma 3.14. *Let Δ be a $(\mathbb{Z}[t], X)$ -graph obtained by a u -folding ϕ from a graph Γ . Let v be a vertex of Γ such that v belongs to some u -component in Γ . Then $\phi(v)$ belongs to a u -component in Δ and $H_u(v) \simeq H_u(\phi(v))$.*

Any finite u -component can be transformed into a single positively oriented path with an associated free abelian group of finite rank. The next results show how one can use u -foldings to get such a form of u -components.

At first, recall that in a connected graph a subgraph is said to be a *spanning tree* if this subgraph is a tree and it contains all vertices of the original graph. If a graph T is a tree then for any two vertices v_1, v_2 of T there is a unique reduced path in T from v_1 to v_2 .

Let Γ be a $(\mathbb{Z}[t], X)$ -graph and $v \in V(\Gamma)$. We call a path $p = e_1 \cdots e_k$ in $\text{Comp}_u(v)$ *positively oriented* (*negatively oriented*) if $\alpha_i > 0$ ($\alpha_i < 0$), $i \in [1, k]$, where $\mu(e_i) = u^{\alpha_i}$.

Lemma 3.15. *Let T be a finite tree such that all its edges are labeled by u^α , $\alpha \in \mathbb{Z}[t]$ and let $v_0 \in V(T)$. Then T can be transformed by finitely many u -foldings into a tree T' such that if $v'_0 \in V(T')$ corresponds to v_0 then for any $v \in V(T')$ the unique reduced path p_v from v'_0 to v is either positively oriented or negatively oriented.*

Proof. Since v_0 can be connected by a unique reduced path to any $v \in V(T)$ and T is finite we have a finite number of such paths p_v , $v \in V(T)$. In each path $p_v = e_1 \cdots e_k$ there can be positive edges, that is, labeled by u^α , $\alpha > 0$ and also negative edges. To prove the statement of the lemma we use induction on the number of vertices in T such that p_v is neither positively nor negatively oriented. Let B_T denote the set of such vertices in T .

If $|B_T| = 0$ then the lemma is already true for T . Thus we assume the statement to be true for any tree S with $|B_S| = n - 1$.

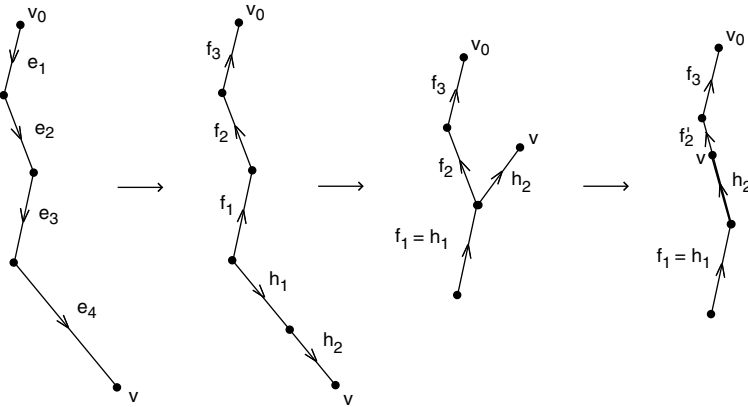


Fig. 2. Case (1), $j = 4$, $i = 2$.

Let $|B_T| = n$ and take any vertex v from B_T . Then $p_v = e_1 \cdots e_k$ is a unique reduced path such that $o(p) = v_0$, $t(p) = v$, $\mu(e_i) = u^{\alpha_i}$, $i \in [1, k]$. Without loss of generality we can assume $\alpha_1 > 0$. Since $v \in B$ there exists minimal $j \in [2, k]$ such that $\alpha_{j-1} > 0$, $\alpha_j < 0$. Thus $t(e_j) \in B_T$ and again, without loss of generality we can assume $v = t(e_j)$. We have two cases.

(1) $\alpha_1 + \cdots + \alpha_{j-1} \geq |\alpha_j|$. There exists maximal $i \in [1, j - 1]$ such that $\alpha_i + \cdots + \alpha_{j-1} > |\alpha_j|$ but $\alpha_{i+1} + \cdots + \alpha_{j-1} < |\alpha_j|$. Hence, e_j can be folded step by step with the negatively oriented path $e_{j-1}^{-1} \cdots e_i^{-1} = f_1 \cdots f_{j-i}$, $e_{j-s}^{-1} = f_s$, $s \in [1, j - i]$ as follows.

e_j is divided into negative edges h_1, \dots, h_{j-i} by new vertices v_1, \dots, v_{j-i-1} so that $o(h_1) = o(e_j)$, $t(h_s) = o(h_{s+1}) = v_s$, $s \in [1, j - i - 1]$, $t(h_{j-i}) = t(e_j)$, $\mu(h_s) = \mu(f_s) = u^{-\alpha_{j-s}}$, $s \in [1, j - i - 1]$, $\mu(h_{j-i}) = u^{\alpha_j + \alpha_{i-1} + \cdots + \alpha_{j-1}}$. Finally, the sequence of u -foldings identifies each h_s with f_s for all $s \in [1, j - i - 1]$ and h_{j-i} becomes an initial part of $f_{j-i} = e_i^{-1}$. We denote this sequence of u -foldings by ϕ and let T' denote the result of applying ϕ to T . Observe that after ϕ is applied $v = t(e_j)$ is identified with a point on edge e_i , so $\phi(v)$ is connected to v_0 by a unique reduced positively oriented path. Obviously T' is also a tree and we have that $\phi(v) \notin B_{T'}$. Observe also that if $w \notin B_T$ then $\phi(w) \notin B_{T'}$. Indeed, if a vertex $w \in T$ is connected to v_0 by a unique path p_w such that $e_j \in p_w$ then p_w also contains $e_1 \cdots e_{j-1}$ and

$w \in B_T$. But such vertices are the only ones for which a path leading to v_0 is changed under ϕ . So if $w' \notin B_T$ then $p_{w'}$ is not affected by ϕ and $\phi(p_{w'}) = p_{w'}$. It follows that $|B_{T'}| < |B_T|$.

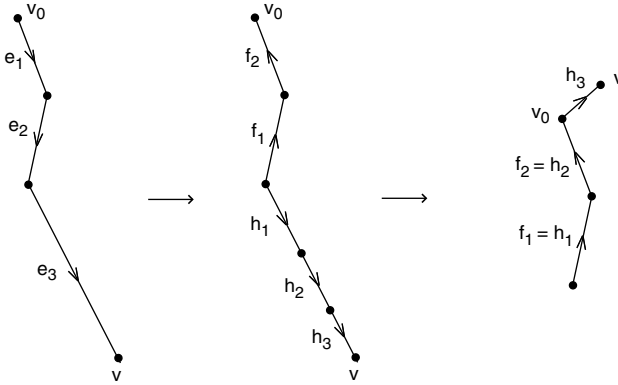


Fig. 3. Case (2), $j = 3$.

(2) $\alpha_1 + \dots + \alpha_{j-1} < |\alpha_j|$. e_j can be folded step by step with the negatively oriented path $e_{j-1}^{-1} \dots e_1^{-1} = f_1 \dots f_{j-1}$, $e_{j-s}^{-1} = f_s$, $s \in [1, j-1]$ as follows.

e_j is divided into negative edges h_1, \dots, h_j by new vertices v_1, \dots, v_{j-1} so that $o(h_1) = o(e_j)$, $t(h_s) = o(h_{s+1}) = v_s$, $s \in [1, j-1]$, $t(h_j) = t(e_j)$, $\mu(h_s) = \mu(f_s) = u^{-\alpha_{j-s}}$, $s \in [1, j-1]$, $\mu(h_{j-1}) = u^{\alpha_j + \alpha_1 + \dots + \alpha_{j-1}}$. Finally, the sequence of u -foldings identifies each h_s with f_s for all $s \in [1, j-1]$. We denote this sequence of u -foldings by ϕ and let T' denote the result of applying ϕ to T . Then T' is also a tree and we have that $\phi(v) = \phi(t(e_j))$ is connected to $\phi(v_0) = v_0$ in T' by a unique reduced path which consists of a single negative edge h_j . So $\phi(v) \notin B_{T'}$ and by the same argument as in (1) we have that $|B_{T'}| < |B_T|$.

Thus, in both cases we obtained a new tree T' for which the statement can be obtained by induction. □

As a corollary of Lemma 3.15 we get the following important result.

Corollary 3.16. *Let T be a finite tree such that all its edges are labeled by u^α , $\alpha \in \mathbb{Z}[t]$ and let $v_0 \in V(T)$. Then T can be transformed into a simple positively oriented path by finitely many u -foldings.*

Proof. By Lemma 3.15, T can be transformed by finitely many u -foldings into a tree T' with $v'_0 \in V(T')$ corresponding to v_0 , such that for any $v \in V(T')$ the unique reduced path p_v from v'_0 to v is either positively oriented or negatively oriented. Now, to complete the proof it is enough to notice that any two finite simple positively oriented paths which have the same origin w can be folded together into one simple positively oriented path with the same origin w . So, after finitely many u -foldings one transforms T' into two paths leading from v'_0 — one positively oriented

and another one negatively oriented. Their concatenation is a reduced positively oriented path. \square

Now, we return to u -components in a $(\mathbb{Z}[t], X)$ -graph Γ .

Lemma 3.17. *Let Γ be a $(\mathbb{Z}[t], X)$ -graph, $v \in V(\Gamma)$ and $C = \text{Comp}_u(v)$ be finite. Then there exist a $(\mathbb{Z}[t], X)$ -graph Δ obtained from Γ by finitely many u -foldings such that $v' \in V(\Delta)$ corresponds to v and $C' = \text{Comp}_u(v')$ consists of a simple positively oriented path $P_{C'}$, some vertices of which may be connected by single edges not in $P_{C'}$.*

Proof. Let T be any spanning tree of C . Then, by Corollary 3.16 there exists a finite sequence $\{\phi_1, \dots, \phi_n\}$ of u -foldings which transforms T into a simple positively oriented path $P_{C'}$. Observe that some vertices in $P_{C'}$ can be connected by images of edges from $C - T$. Thus, Δ is the image of Γ under $\{\phi_1, \dots, \phi_n\}$. \square

We call C' from the lemma above a *reduced u -component*. Since $P_{C'}$ is a simple path there exists a vertex $z_{C'} \in V(P_{C'})$ such that $\text{val}_{P_{C'}}(z_{C'}) = 1$ and the only edge in $P_{C'}$ which has $z_{C'}$ as an origin is positive. We call $z_{C'}$ a *base-point* of C' .

Observe that because of arbitrary choice of a spanning tree for C and arbitrary order of performing sequences of u -foldings it follows that the reduced u -component C' corresponding to C is not unique.

Let Γ be any $(\mathbb{Z}[t], X)$ -graph, $v \in V(\Gamma)$ and let $C = \text{Comp}_u(v)$ be a finite reduced u -component. Then there exists a simple positively oriented path P_C in C which originates from a base-point z_C of C . Since C is finite then there are finitely many edges h_1, \dots, h_l in $C - P_C$. Any h_i connects two vertices in P_C so there exists a unique reduced path q_i in P_C such that $o(h_i) = o(q_i)$, $t(h_i) = t(q_i)$. Moreover, $h_i q_i^{-1}$ is a cycle in C , so $\mu(h_i) * \overline{\mu(q_i)}^{-1} = c \in H_u(z_C)$.

Now, let $p = e_1 \cdots e_n$ be a reduced path in C . Suppose some of its edges e_{i_1}, \dots, e_{i_k} , $i_j \in [1, n]$ belong to $C - P_C$. Then we construct another path p' in the following way: every edge e_{i_j} is equal to some h_{i_j} in the list of edges of $C - P_C$, so in p we substitute e_{i_j} by the path q_{i_j} . The resulting path p' may be not reduced, so we perform all possible reductions and obtain a reduced path p'' all edges of which belong to P_C . We have $o(p'') = o(p)$, $t(p'') = t(p)$, $\overline{\mu(p'')} = \overline{\mu(p)} * h$, $h \in H_u(z_C)$ by definition of q_i for each $h_i \in C - P_C$. Finally, observe that p'' is uniquely defined for p because it is a reduced subpath of P_C .

Hence, for any reduced path p in C there exists a unique reduced subpath q of P_C with the same endpoints as p and such that $\overline{\mu(p)} * \overline{\mu(q)}^{-1} \in H_u(z_C)$. Further, we will use the notation $q = [p]$.

The converse is also true, that is, if q is a reduced path in P_C and $c \in H_u(z_C)$ then, since any element from $H_u(z_C)$ can be realized as a reduced label of some loop at $t(q)$ in C , there exists a reduced path p in C such that $o(p) = o(q)$, $t(p) = t(q)$, $\overline{\mu(p)} = \overline{\mu(q)} * c$. Observe that q is not unique with respect to permutation of cycles.

The correspondence above shows that any finite reduced u -component C in a $(\mathbb{Z}[t], X)$ -graph is characterized completely by the pair $(P_C, H_u(z_C))$.

If C is finite then there exist finitely many subpaths q_1, \dots, q_s of P_C such that for any path p in C , $[p] = q_i$ for some $i \in [1, s]$. Moreover, let $P_C = f_1 \cdots f_m$, where $o(f_1) = z_C$. Let $v_0 = z_C$, $v_i = t(f_i)$, $i \in [1, m]$ and let p_0, p_1, \dots, p_m be reduced subpaths of P_C such that $o(p_i) = z_C$, $t(p_i) = v_i$, $i \in [0, m]$. It follows that all p_i are positively oriented. Then for every reduced subpath $p_{i,j}$ of P_C such that $o(p_{i,j}) = v_i$, $t(p_{i,j}) = v_j$ we have $\overline{\mu(p_{i,j})} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}$. By Lemma 3.11, $H_u(z_C)$ is finitely generated and is isomorphic to a subgroup of \mathbb{Z}^r , $r \in \mathbb{N}$. So if p is a reduced path in C such that $o(p) = v_i$, $t(p) = v_j$ then by definition of $[p]$ we have $\overline{\mu(p)}$ and $\overline{\mu([p])} = \overline{\mu(p_{i,j})} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}$ are in the same coset in \mathbb{Z}^r by $H_u(z_C)$. That is, we can express the label of any reduced path in C in terms of labels of p_i , $i \in [0, m]$ and elements from $H_u(z_C)$. We call a set of paths p_0, p_1, \dots, p_m a set of path representatives associated with C and denote this set by $Rep(C)$. The following result holds.

Lemma 3.18. *Let C be a finite reduced u -component in a $(\mathbb{Z}[t], X)$ -graph Γ , $v \in V(C)$ and let $\alpha \in \mathbb{Z}[t]$. If $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} \notin H_u(z_C)$ for any $p_i, p_j \in Rep(C)$, $i \neq j$ then either there exists a unique reduced path p in P_C such that $o(p) = v$ and $u^\alpha \in \overline{\mu(p)} * H_u(z_C)$, or there exists no path q in C with this property.*

Proof. Suppose on the contrary that there exist two reduced paths p, q in P_C such that $o(p) = v$, $o(q) = v$ and $u^\alpha = \overline{\mu(p)} * h_1 = \overline{\mu(q)} * h_2$, $h_1, h_2 \in H_u(z_C)$. Then $\overline{\mu(p)} * \overline{\mu(q)}^{-1} = h_1^{-1} * h_2 \in H_u(z_C)$.

On the other hand we have that $v = v_i$, $t(p) = v_j$, $t(q) = v_k$, $j \neq k$. So we have $\overline{\mu(p)} = \overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}$, $\overline{\mu(q)} = \overline{\mu(p_k)} * \overline{\mu(p_i)}^{-1}$. So $\overline{\mu(p)} * \overline{\mu(q)}^{-1} = (\overline{\mu(p_j)} * \overline{\mu(p_i)}^{-1}) * (\overline{\mu(p_k)} * \overline{\mu(p_i)}^{-1})^{-1} = \overline{\mu(p_j)} * \overline{\mu(p_k)}^{-1} \in H_u(z_C)$ — contradiction. \square

If C is reduced and $Rep(C)$ satisfies the condition from Lemma 3.18 then we call C a u -folded u -component.

Let C be a finite reduced u -component in a $(\mathbb{Z}[t], X)$ -graph Γ which is not u -folded. That is, there exist two vertices v_i, v_j , $i < j$ in P_C such that $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} \in H_u(z_C)$. Consider a graph Δ which is obtained from Γ by identification of vertices v_i, v_j in C into one new vertex v . We call this operation a collapse of v_i and v_j . The resulting u -component in Δ we denote by C' . In fact, a collapse can be obtained as a finite sequence of u -foldings. Indeed, since $\overline{\mu(p_i)} * \overline{\mu(p_j)}^{-1} = h \in H_u(z_C)$ there exists a positive loop at v_i with the label h . This means that if we add a single edge e to C so that $o(e) = t(e) = v_i$, $\mu(e) = h$ then $H_u(z_C)$ is not changed. Then we can apply a sequence of $j - i$ u -foldings to e and the subpath $q = e_{i+1} \cdots e_j$ of P_C connecting $\overline{v_i}$ and v_j . After these u -foldings are implemented, v_i is identified with v_j because $\overline{\mu(q)} = h$. Since u -foldings do not change $H_u(z_C)$ in u -components a collapse of v_i and v_j is a valid operation. Observe that $|V(C)| = |V(C')| + 1$. If C' is not reduced so using finitely many u -foldings

one can reduce it and since u -foldings do not increase the number of vertices for the resulting reduced u -component C'' we have $|V(C)| > |V(C')| \geq |V(C'')|$. Thus, the following result holds.

Lemma 3.19. *Let Γ be a $(\mathbb{Z}[t], X)$ -graph which has finitely many u -components all of which are finite and reduced. Then there exists a partially folded $(\mathbb{Z}[t], X)$ -graph Δ which is obtained from Γ by finitely many u -foldings such that all its u -components are u -folded.*

4. U -Folded $(\mathbb{Z}[t], X)$ -Graphs and Finitely Generated Subgroups of $F^{\mathbb{Z}[t]}$

In this section we introduce the notion of U -folded $(\mathbb{Z}[t], X)$ -graph. Using the notations developed in the previous section we show how one can transform an arbitrary $(\mathbb{Z}[t], X)$ -graph into a U -folded one.

4.1. Languages associated with $(\mathbb{Z}[t], X)$ -graphs

Recall that a graph labeled by letters from X^\pm defines a language of words over X^\pm . This language can be put into correspondence with a subgroup of a free group $F(X)$. In the present subsection we generalize this concept to $(\mathbb{Z}[t], X)$ -graphs.

Definition 4.1. Let Γ be a $(\mathbb{Z}[t], X)$ -graph and let v be a vertex of Γ . We define the language of Γ with respect to v to be

$$L(\Gamma, v) = \{\overline{\mu(p)} \mid p \text{ is a reduced path in } \Gamma \text{ from } v \text{ to } v\}.$$

If w belongs to $L(\Gamma, v)$, we will also sometimes say that w is *accepted* by (Γ, v) (or just by Γ if v is fixed).

The following result establishes a connection between $(\mathbb{Z}[t], X)$ -graphs and subgroups in $F^{\mathbb{Z}[t]}$.

Lemma 4.2. *Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph and let $v \in V(\Gamma)$. Then $L(\Gamma, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$.*

Proof. Observe, at first, that $L(\Gamma, v)$ is a subset of $F^{\mathbb{Z}[t]}$ by definition.

Let $g_1, g_2 \in L(\Gamma, v)$. Then there are reduced paths p_1 and p_2 from v to v in Γ such that $\overline{\mu(p_i)} = g_i, i = 1, 2$.

The concatenation q of p_1 and p_2 is a path in Γ from v to v such that $\overline{\mu(q)} = \overline{\mu(p_1) * \mu(p_2)}$, but q may not be reduced. Let p be the reduced path obtained from q by making all possible path reductions. This means that the label $\overline{\mu(p)} = \overline{\mu(q)}$ and $o(p) = t(p) = v$. Therefore $\overline{\mu(p_1) * \mu(p_2)} = \overline{\mu(p)} \in L(\Gamma, v)$.

Thus, $g_1 * g_2 \in L(\Gamma, v)$ and $L(\Gamma, v)$ is closed under multiplication $*$ of infinite words.

It is easy to see that the inverse path $(p_1)^{-1}$ of p_1 is reduced and $\overline{\mu(p_1^{-1})} = \overline{\mu(p_1)^{-1}} = \overline{\mu(p_1)}^{-1}$. This implies that $L(\Gamma, v)$ is closed under taking inverses. Also, obviously $\varepsilon \in L(\Gamma, v)$.

Thus, $L(\Gamma, v)$ is a subgroup of $F^{\mathbb{Z}[t]}$. □

In the previous section we introduced free foldings and U -foldings and now we show that these operations do not change the language associated with Γ .

Lemma 4.3. *Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph and let $v \in V(\Gamma)$. Let Δ_1 be a $(\mathbb{Z}[t], X)$ -graph obtained from Γ by a single free folding and let Δ_2 be a $(\mathbb{Z}[t], X)$ -graph obtained from Γ by a single u -folding for some $u \in U$, so that $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ correspond to v . Then*

$$L(\Gamma, v) = L(\Delta_1, v_1) = L(\Delta_2, v_2).$$

Proof. At first we prove $L(\Gamma, v) = L(\Delta_1, v_1)$. If Δ_1 is obtained from Γ by identification of edges $e_1, e_2 \in \widehat{\Gamma}$, $\mu(e_1) = \mu(e_2) = x \in X^{\pm 1}$ then the result follows from the proof of [4, Lemma 3.4]. If $e_1, e_2 \in \widehat{\Gamma}$ are labeled by the same exponent of some $u \in U$ then the free folding which identifies e_1 and e_2 is a u -folding and, thus, in this case, we reduced the proof of $L(\Gamma, v) = L(\Delta_1, v_1)$ to the proof of $L(\Gamma, v) = L(\Delta_2, v_2)$.

So let us prove $L(\Gamma, v) = L(\Delta_2, v_2)$.

Suppose Δ_1 is obtained from Γ by folding two edges e_1, e_2 in $\widehat{\Gamma}$ which have the same initial vertex w and $\mu(e_1) = u^\alpha$, $\mu(e_2) = u^\beta$. Without loss of generality we can assume $\alpha \geq \beta > 0$. Then e_1 becomes a path $h_1 h_2$ in Δ_2 such that $\mu(h_1) = u^\beta$, $\mu(h_2) = u^{\alpha-\beta}$ and h_1 is identified with e_2 in Δ_2 into an edge h .

Suppose p is a reduced path in Γ from v to v , so that $\overline{\mu(p)} \in \underline{L(\Gamma, v)}$. The image of p in Δ_1 by Lemma 3.13 is a path p' from v_2 to v_2 such that $\overline{\mu(p)} = \overline{\mu(p')}$. However, p' need not be reduced. Namely, p' is reduced if and only if p does not contain any subpaths of the form $e_2^{-1}e_1$ or $e_1^{-1}e_2$. Let p'' be the path obtained from p' by performing all possible path reductions in Δ_2 . Then $\overline{\mu(p)} = \overline{\mu(p')} = \overline{\mu(p'')}$ and $\overline{\mu(p'')} \in L(\Delta_2, v_2)$. Thus, we have shown that $L(\Gamma, v) \subseteq L(\Gamma', v')$.

Suppose now that p' is an arbitrary reduced path in Δ_2 from v_2 to v_2 . We claim that there is a reduced path p in Γ from v to v such that $\overline{\mu(p)} = \overline{\mu(p')}$. We will construct this path explicitly.

(1) $\alpha = \beta$. In this case h_2 is an empty edge and e_1, e_2 are identified with the edge h in Δ_2 .

The occurrences of $h^{\pm 1}$ (if any) subdivide p' into a concatenation of the form:

$$p' = p_0 f_0 p_1 f_1 \cdots f_k p_{k+1},$$

where $f_i = h^{\pm 1}$ and the paths p_i do not involve $h^{\pm 1}$.

Suppose that for some i we have $f_i = h$. Since p_i and p_{i+1} do not involve the edge h , they can also be considered as paths in Γ . Moreover, by the definition of u -folding, in the graph Γ the terminal vertex of p_i is joined with the initial vertex of p_{i+1} by either the edge e_1 or e_2 . We denote this edge by d_i (so that $d_i \in \{e_1, e_2\}$). Note that now $p_i d_i p_{i+1}$ is a reduced path in Γ with the same label as the path $p_i f_i p_{i+1}$ in Δ_2 .

Similarly, if for some i we have $f_i = h^{-1}$, we can find $d_i \in \{e_1^{-1}, e_2^{-1}\}$ such that $p_i d_i p_{i+1}$ is a reduced path in Γ with the same label as the path $p_i f_i p_{i+1}$ in Δ_2 .

Then

$$p = p_0 d_0 p_1 \cdots d_k p_{k+1}$$

is a reduced path in Γ from v to v with the same label as p' . Thus, $\overline{\mu(p')} \in L(\Gamma, v)$ and therefore $L(\Delta_2, v_2) \subseteq L(\Gamma, v)$.

(2) $\alpha > \beta$. This is a general case when e_1 becomes a path hh_2 and e_2 becomes the edge h in Δ_2 . Observe that $\mu(h) = \mu(e_2)$, $\mu(h_2) = \mu(e_2)^{-1} * \mu(e_1)$ and $o(h) = o(e_2) = o(e_1)$, $t(h) = t(e_2)$, $o(h_2) = t(h) = t(e_2)$, $t(h_2) = t(e_1)$ in Δ_2 .

Similarly to (1) we subdivide p' by the occurrences of $h^{\pm 1}$ and $h_2^{\pm 1}$ so that

$$p' = p_0 f_0 p_1 f_1 \cdots f_k p_{k+1},$$

where $f_i = h^{\pm 1}$ or $f_i = h_2^{\pm 1}$ and the paths p_i do not involve $h^{\pm 1}$, $h_2^{\pm 1}$.

Any entry of edge h_2 we substitute by a path $e_2^{-1}e_1$ and every entry of h we substitute by e_2 so that h_2^{-1} is substituted by $e_1^{-1}e_2$ and h^{-1} by e_2^{-1} .

Suppose that for some i we have $f_i = h$. Since p_i and p_{i+1} do not involve the edge h , they can also be considered as paths in Γ . We have $t(p_i) = o(h) = o(e_2)$, $o(p_{i+1}) = t(h) = t(e_2)$, so the substitution $h \rightarrow e_2$ is valid. If on the other hand for some j we have $f_j = h_2$ then $t(p_j) = o(h_2) = t(e_2)$, $o(p_{j+1}) = t(h_2) = t(e_1)$ and again the substitution $h_2 \rightarrow e_2^{-1}e_1$ is valid. Finally, $\mu(h) = \mu(e_2)$, $\mu(h_2) = \mu(e_2)^{-1} * \mu(e_1)$ implies that after all possible substitutions in p' are made, we get the resulting path p'' in Γ from v to v such that $\overline{\mu(p'')} = \overline{\mu(p')}$, but p'' may not be reduced. Let p be the path obtained from p'' by performing all possible path reductions in Γ . Then $\overline{\mu(p)} = \overline{\mu(p'')} = \overline{\mu(p')}$ and $\overline{\mu(p)} \in L(\Gamma, v)$.

Hence, in both cases (1) and (2) above $L(\Delta_2, v_2) \subseteq L(\Gamma, v)$ which completes the proof. \square

Let Γ be a $(\mathbb{Z}[t], X)$ -graph and $p = e_1 \cdots e_k$ be a reduced path in Γ . Let $g \in G_{n+1} - G_n$ and let

$$\pi(g) = \pi(h_1)u^{\beta_1}\pi(h_2) \cdots u^{\beta_l}\pi(h_{l+1}),$$

be the standard decomposition of g , where $u = \max\{U(g)\}$. We write

$$\mu(p) = \pi(g)$$

if p can be subdivided into subpaths

$$p = p_1 d_1 p_2 \cdots d_l p_{l+1},$$

where d_i is a path in some u -component of Γ and p_i is a path in Γ which does not contain edges labeled by u^α , $\alpha \in \mathbb{Z}[t]$, so that $\overline{\mu(d_i)} = u^{\beta_i}$, $i \in [1, l]$ and each equality $\mu(p_i) = \pi(h_i)$, $i \in [1, l+1]$ is defined inductively in the same way. Observe that if $g = x_1 \cdots x_r \in F$ then $\mu(p) = \pi(g)$ if $k = r$ and $\mu(e_i) = x_i$ for every $i \in [1, k]$.

It follows immediately that if $\mu(p) = \pi(g)$ for some $g \in F^{\mathbb{Z}[t]}$ then p is label reduced.

Observe that if Γ is a $(\mathbb{Z}[t], X)$ -graph and $v \in V(\Gamma)$ then it follows from Lemma 4.2 that any element of $L(\Gamma, v)$ is an infinite word in $F^{\mathbb{Z}[t]}$ and, thus, has

the standard decomposition $\pi(g)$ but it might be impossible to find a label reduced loop p at v in Γ such that $\mu(p) = \pi(g)$. The main result of the following subsection is that if Γ is finite and has some particular properties then it is always possible to find such a path p and p is unique in some sense. Moreover, it will be shown how the required properties can be obtained in Γ using free foldings and U -foldings.

4.2. U -folded $(\mathbb{Z}[t], X)$ -graphs

In this subsection we obtain the main technical result of the paper.

Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph. Since Γ is finite there exist only finitely many edges with labels u_e^α , $u_e \in U$, $\alpha \in \mathbb{Z}[t]$. Thus, there exists $K \in \mathbb{N}$ such that for any $e \in E(\Gamma)$ with $\mu(e) = u_e^\alpha$ it follows that $u_e \in U_{j_e}$, $j_e \leq K$. It is easy to see that for each fixed j_e using u_e -foldings described in Sec. 3 one can transform all u_e -components of Γ into u_e -folded components. However, because of the nature of the sets U_{j_e} , edges of Γ belong to different *levels* (we introduce the precise definition of a level below) and U -foldings do not deal with interactions between these levels. Moreover, u -foldings applied to some u - component can affect w -components $u \neq w$. So, one needs some definite procedure which “folds” Γ level by level.

Since Γ is finite the set of elements $u \in U$ such that there exists an edge e in Γ labeled by u^α , $\alpha \in \mathbb{Z}[t]$ is finite and the order from U is induced on it. Thus one can associate with Γ an ordered set $U(\Gamma) = \{u_1, \dots, u_N\}$, where $N > 0$, $u_i \in U$, $i \in [1, N]$ and $u_i < u_j$ if $i < j$. Observe that $U(\Gamma)$ can be empty in case when all edges of Γ are labeled by letters from X^\pm .

Definition 4.4. Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph and let $u_i \in U(\Gamma)$ be fixed. Let $\Gamma(i)$ be a subgraph of Γ which consists only of edges $e \in E(\Gamma)$ such that either $\mu(e) = w \in X^\pm$ or $\mu(e) = u_j^\alpha$, $\alpha \in \mathbb{Z}[t]$, $j \leq i$. We call $\Gamma(i)$ an *i -level graph* of Γ (by 0-level graph we understand a subgraph of Γ which consists only of edges with labels from X) and say that Γ has *level n* denoted by $l(\Gamma)$ if n is the minimal natural number for which $\Gamma = \Gamma(n)$.

Observe that $\Gamma(i)$ may be not connected for some $i < l(\Gamma)$, but still one can apply to $\Gamma(i)$ free foldings and u -foldings $u \in U(\Gamma)$.

Definition 4.5. Let Γ be a finite connected $(\mathbb{Z}[t], X)$ -graph and let $U(\Gamma) = \{u_1, \dots, u_{l(\Gamma)}\}$. If $u_n \in U(\Gamma)$ and C is a u_n -component of Γ then

- (1) a path p in $\Gamma(n-1)$ is called *(u_n, δ) -irregular of type I* if $o(p) \in V(C)$, $\overline{\mu(p)} = w$, $w = u_n^\delta \circ c_p$, $\delta \in \{1, -1\}$,
- (2) a path p in $\Gamma(n)$ is called *(u_n, δ) -irregular of type II* if $p = p_{\Gamma(n-1)} p_{\Gamma(n)}$, $o(p) \in V(C)$, $p_{\Gamma(n-1)} \in \Gamma(n-1)$, $\overline{\mu(p_{\Gamma(n-1)})} = w_1$, $\overline{\mu(p_{\Gamma(n)})} = w_2 \circ c = u_n^\gamma \circ c_1$, $\gamma \in \mathbb{Z}[t] - \mathbb{Z}$, $u_n^\delta = w_1 \circ w_2$, $w_2 \neq \varepsilon$, $\delta \in \{1, -1\}$,
- (3) a path p in $\Gamma(n-1)$ is called *(u_n, δ) -irregular of type III* if $o(p) \in V(C)$, $\overline{\mu(p)} = w_1(p)$, $u_n^\delta = w_1(p) \circ w_2(p)$, $\overline{\mu_2(p)} \neq \varepsilon$, $\delta \in \{1, -1\}$ and there exists a path p' in C such that $o(p') = o(p)$, $\overline{\mu(p')} = u_n^\gamma$, $\gamma \in \mathbb{Z}[t]$, $\gamma\delta > 0$.

Definition 4.6. Let Γ be a finite connected $(\mathbb{Z}[t], X)$ -graph and let $U(\Gamma) = \{u_1, \dots, u_{l(\Gamma)}\}$. If $u_n \in U(\Gamma)$ and C is a u_n -component of Γ then

- (1) a (u_n, δ) -irregular path p of type I originating from v is *doubled* if there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v$, $t(q) = t(q_2) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)}$, $\mu(q_1) = \pi(u_n)^\delta$, $t(q_1) \in V(C)$,
- (2) a (u_n, δ) -irregular path p of type II originating from v is *doubled* if there exists a label reduced path $q = q_1 q_2$ in $\Delta(n)$ such that $o(q) = v$, $t(q) = t(q_2) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)}$, $\mu(q_1) = \pi(u_n)^\delta$, $t(q_1) \in V(C)$,
- (3) a (u_n, δ) -irregular path p of type III originating from v is *doubled* if there exists a label reduced path $q = q_1 q_2$ in $\Delta(n-1)$ such that $o(q) = v$, $t(q) = t(q_2) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)}$, $\mu(q_1) = \pi(w_1(p))$, $\mu(q_2) = \pi(w_2(p))$, $t(q) \in V(C)$.

Definition 4.7. Let Δ be a finite connected $(\mathbb{Z}[t], X)$ -graph and let $U(\Delta) = \{u_1, \dots, u_{l(\Delta)}\}$. Δ is called *U-folded* if for any $u_n \in U(\Delta)$ the following conditions are satisfied:

- (1) Δ is partially folded;
- (2) all u_n -components of Δ are u_n -folded and *isolated* in the sense that there exists no reduced path p with $\overline{\mu(p)} = u_n^k$, $k \in \mathbb{Z}$ in $\Delta(n-1)$ such that p connects two different u_n -components of Δ ;
- (3) if C is a u_n -component of Δ , $e \in E(P_C)$ and $\mu(e) = u_n^k$, $k \in \mathbb{Z}$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = o(e)$, $t(p) = t(e)$, $\mu(p) = \pi(u_n)^k$;
- (4) if C is a u_n -component of Δ and $v \in V(C) \cap V(\Delta(n-1))$ then there exists a unique label reduced path p in $\Delta(n-1)$ such that $o(p) = t(p) = v$, $\mu(p) = \pi(u_n)^k$, $k \in \mathbb{Z}$ and $H_{u_n}(z_C) \cap \langle u_n \rangle = \langle u_n^k \rangle$;
- (5) if C is a u_n -component of Δ and $v_1, v_2 \in V(C)$ are connected by a reduced subpath p of P_C then either p consists only of edges labeled by finite exponents of u_n or there exists no number $k_p \in \mathbb{Z}$ such that $\overline{\mu(p)} * u_n^{-k_p} \in H_{u_n}(z_C)$;
- (6) for any u_n -component C of Δ and its distinct vertices v_1, v_2 which are connected by a reduced subpath p of P_C with $o(p) = v_1$, $t(p) = v_2$ there exists no reduced path r in $\Delta(n-1)$ such that $o(r) = v_1$, $t(r) = v_2$, $\overline{\mu(r)} = u_n^k$, $k \in \mathbb{Z}$ and $\overline{\mu(p)} * \overline{\mu(r)}^{-1} \notin H_{u_n}(v_1)$;
- (7) for any u_n -component C of Δ , $v \in V(C)$ and a reduced path p in $\Delta(n-1)$ such that $o(p) = v$, $\overline{\mu(p)} = u_n^k$, $k \in \mathbb{Z}$ it follows that $t(p) \in V(C)$;
- (8) for any u_n -component C of Δ and its vertex v any (u_n, δ) -irregular path p of any type originating from v is doubled in $\Delta(n)$.

Remark 4.8. Observe that in (3) and (4) by uniqueness we understand uniqueness with respect to P_C , where C is any u -component $u \in U(\Delta)$, that is, a path is unique if we disregard the order of edges in $C - P_C$. This is justified in view of Lemma 3.18.

The definition above is very technical and it is not clear why we need all the properties listed. However, using them in the proof of the next proposition we are able to show the most essential properties of U -folded graphs.

Proposition 4.9. *Let Δ be a U -folded graph. Then the following hold:*

- (1) *for any reduced path p in Δ with $\overline{\mu(p)} = w$ there exists a unique label reduced path q such that $o(q) = o(p)$, $t(q) = t(p)$, $\mu(q) = \pi(w)$;*
- (2) *for the standard decomposition $\pi(g)$ of any $g \in F^{\mathbb{Z}[t]}$ and any $v \in V(\Delta)$, either there exists a unique label reduced path p in Δ starting at v such that $\mu(p) = \pi(g)$ or for any path q in Δ starting at v it follows that $\overline{\mu(q)} \neq g$.*

Proof. We take advantage of the level structure of Δ — the proof is conducted by the induction on $l(\Delta)$. If $l(\Delta) = 0$, that is, all edges in Δ are labeled by letters from X^\pm , then Δ is partially folded and the statement of the proposition follows from [4, Lemma 3.9].

Assume that the statement of the theorem holds for any graph of level $n - 1$. Let $l(\Delta) = n$ and let $U(\Delta) = \{u_1, \dots, u_{l(\Delta)}\}$.

(1) Let p be a path in Δ with $\overline{\mu(p)} = w$. $\mu(p)$ belongs to the alphabet $X^\pm \cup \{u^\alpha \mid u \in U(\Delta), u \leq u_n, \alpha \in \mathbb{Z}[t]\}$ so let us subdivide p in the following way:

$$p = p_1 d_1 p_2 d_2 \cdots p_m d_m p_{m+1},$$

where every p_i , $i \in [1, m + 1]$ does not contain edges labeled by u_n^α , $\alpha \in \mathbb{Z}[t]$, so that we can assume every p_i to be a path in $\Delta(n - 1)$, and every d_i , $i \in [1, m]$ contains only edges labeled by u_n^α , $\alpha \in \mathbb{Z}[t]$.

Take any p_i . By induction there exists a unique label reduced path q_i such that $o(p_i) = o(q_i)$, $t(p_i) = t(q_i)$, $\mu(q_i) = \pi(h_i)$, where $\overline{\mu(p_i)} = h_i$. Observe that since $t(p_i)$ belongs to some u_n -component C_i and d_i is a path composed only of edges labeled by exponents of u_n then d_i is a path in C_i such that $\overline{\mu(d_i)} = u_n^{\alpha_i}$. By Lemma 3.18 there exists a unique path c_i in P_{C_i} such that $o(c_i) = o(d_i)$ and $u_n^{\alpha_i} \in \overline{\mu(c_i)} * H_{u_n}(o(d_i))$. We have $t(d_i) = t(c_i)$. Indeed, if d_i leads to some other vertex $v \in P_{C_i}$ then there exists a unique positively oriented subpath p_v of P_{C_i} from $o(d_i)$ to v and we have a loop $\overline{p_v d_i^{-1}}$, so $\overline{\mu(p_v)} * u_n^{-\alpha_i} \in H_{u_n}(o(d_i))$. Thus, $u_n^{\alpha_i} = \overline{\mu(p_v)} * h_i$, $h_i \in H_{u_n}(o(d_i))$ and $\overline{\mu(p_v)} \in \overline{\mu(c_i)} * H_{u_n}(o(d_i))$ which is impossible unless $p_v = c_i$, because C_i is u_n -reduced. So $t(d_i) = t(c_i) = o(p_{i+1})$.

Thus, we obtain a reduced path r as the following concatenation:

$$q' = q_1 c_1 q_2 c_2 \cdots q_m c_m q_{m+1},$$

where $o(q_i) = o(p_i)$, $t(q_i) = t(p_i)$, $o(c_i) = o(d_i)$, $t(c_i) = t(d_i)$, $\overline{\mu(q_i)} = \overline{\mu(p_i)}$, $\overline{\mu(c_i)} = \overline{\mu(d_i)} = u_n^{\alpha_i}$, $i \in [1, m + 1]$ and $o(q') = o(p)$, $t(q') = t(p)$, $\overline{\mu(q')} = \overline{\mu(p)}$ and all q_i, c_i are label reduced.

However, q' may be not label reduced, that is, there can be a cancellation in $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})}$. Without loss of generality we can assume $\alpha_i \gg 0$ (if not then by the

property (3) of U -folded graphs we can assume $p_i c_i p_{i+1}$ to be a path in $\Delta(n - 1)$ and everything follows by induction for $p_i c_i p_{i+1}$.

Suppose $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})} \neq \overline{\mu(c_i)} \circ \overline{\mu(q_{i+1})}$. Then $\overline{\mu(q_{i+1})}$ contains an initial subword $u_n^k \circ g_1$, $k < 0$, $u_n^{-1} = g_1 \circ g_2$. $\Delta(n - 1)$ is U -folded and the induction hypothesis holds for it, hence it follows that there exists a path $r_{i+1} = z_{i+1} f_{i+1}$ such that $o(r_{i+1}) = o(q_{i+1})$, $t(r_{i+1}) = t(q_{i+1})$, $o(z_{i+1}) = v_1 \in V(C_i)$, $\mu(z_{i+1}) = \pi(u_n)^k \pi(g_1)$ and it can be continued by a path z' to some vertex $v_2 \in V(C_i)$ so that $z_{i+1} z'$ is a label reduced path from v_1 to v_2 . Let w_{i+1} be the path in P_{C_i} such that $o(w_{i+1}) = v_1$, $t(w_{i+1}) = v_2$. Then we have that concatenation $w_{i+1} z'^{-1}$ is a path from $t(c_i) = o(q_{i+1}) = o(z_{i+1})$ to $t(z_{i+1}) = o(f_{i+1})$ and $\overline{\mu(w_{i+1} z'^{-1})} = \overline{\mu(z_{i+1})} = u_n^k \circ g_1$. So we can substitute q_{i+1} by $w_{i+1} z'^{-1} f_{i+1}$ in q . Observe that $\mu(z'^{-1}) = g_2^{-1}$ and $z'^{-1} f_{i+1}$ is label reduced otherwise $u_n^k \circ g_1$ is not the maximal initial subword of $\overline{\mu(q_{i+1})}$ which cancels in $\overline{\mu(c_i)} * \overline{\mu(q_{i+1})}$. Also, $u_n * g_2^{-1} = u_n \circ g_2^{-1}$.

Thus, $c_i q_{i+1}$ becomes $c_i w_{i+1} z'^{-1} f_{i+1}$. We can find a unique label reduced path b_i in C_i which corresponds to $c_i w_{i+1}$ and then $b_i z'^{-1} f_{i+1}$ is label reduced.

If $\overline{\mu(q_i)} * \overline{\mu(c_i)} \neq \overline{\mu(q_i)} \circ \overline{\mu(c_i)}$ then we can substitute q_i by $f_i z''^{-1} w_i$ and using the same argument as above show that we obtain a label reduced path $f_i z''^{-1} b_i$ which corresponds to $q_i c_i$.

After finitely many such substitutions we get a label reduced path q such that $o(q) = o(p)$, $t(q) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)} = w$. Finally, using the property (8) one can construct a unique label reduced path q' such that $o(q') = o(p)$, $t(q') = t(p)$, $\mu(q') = \pi(w)$.

(2) Let $g \in F^{\mathbb{Z}[t]}$ be such that

$$\pi(g) = \pi(h_1) u^{\alpha_1} \pi(h_2) \cdots u^{\alpha_m} \pi(h_{m+1}).$$

If $u < u_n$ then (2) follows by induction. Suppose $u = u_n$. By induction either there exists a unique label reduced path p_1 in $\Delta(n - 1)$ for h_1 originating from $v \in \Delta(n)$ such that $\overline{\mu(p_1)} = \pi(h_1)$ or for any path q_1 in Δ_7 originating from v it follows that $\overline{\mu(q_1)} \neq h_1$. In the latter case there exists no path p for g such that $\overline{\mu(p)} = g$, because if it exists then by (1) there exists a unique path q such that $o(q) = o(p)$, $t(q) = t(p)$, $\mu(q) = \pi(g)$. But q contains an initial subpath q_1 originating from v with the label $\mu(q_1) = \pi(h_1)$ — a contradiction. In the former case we continue with u^{α_1} . Since $t(p_1)$ belongs to some u -component C_1 of $\Delta(n)$ then by Lemma 3.18 either there exists a unique path c_1 in P_{C_1} such that $o(c_1) = t(p_1)$ and $u^{\alpha_1} \in \overline{\mu(c_1)} * H_u(o(c_1))$ or there exists no continuation of p_1 in C_1 which is labeled by u^{α_1} . Again, if this continuation does not exist then there exists no required path for g , if it does exist then we continue. After at most $2m + 1$ number of steps we get the required result. This completes the proof of (2) and the proposition. □

In order to prove the main technical result of this section we need several lemmas.

Lemma 4.10. *Let Γ be a finite U -folded $(\mathbb{Z}[t], X)$ -graph such that $U(\Delta) = \{u_1, \dots, u_{l(\Delta)}\}$ and let $u \in U$ be such that $u > u_{l(\Delta)}$. If $v \in V(\Gamma)$ and there exists a path p in Γ such that $o(p) = v$, $\mu(p) = \pi(u^{\pm k})$, $k \in \mathbb{N}$ then there exists a path q in Γ such that $o(q) = v$, $\mu(q) = \pi(u^{\pm 1})$.*

Proof. By the choice of U we have $\pi(u^2) = \pi(u) \pi(u) = \pi(u)^2$. Hence, it implies that a path in Γ originating from v and labeled by $\pi(u^k)$, $k \in \mathbb{N}$ contains an initial subpath labeled by $\pi(u)$.

On the other hand, let p be a path in Γ such that $o(p) = v$, $\mu(p) = \pi(u^{-k})$, $k \in \mathbb{N}$. Thus, we have a path p^{-1} originating from $t(p)$ such that $\overline{\mu(p^{-1})} = u^k$. Since Γ is U -folded it follows that there exists a path r in Γ such that $o(r) = t(p)$, $\mu(r) = \pi(u^k)$. By the choice of U we have $\pi(u^k) = \pi(u)^k$ and hence r contains a terminal subpath r_1 such that $t(r_1) = v$, $\mu(r_1) = \pi(u)$. Thus, for r_1^{-1} we have $o(r_1^{-1}) = v$, $\overline{\mu(r_1^{-1})} = u^{-1}$. Finally, since Γ is U -folded there exists a path q such that $o(q) = v$, $\mu(q) = \pi(u^{-1})$. □

Lemma 4.11. *Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph such that $U(\Gamma) = \{u_1, \dots, u_{l(\Gamma)}\}$. Let $u_n \in U(\Gamma)$, C be a u_n -component of Γ , $v \in V(C)$. Suppose $\Gamma(n-1)$ is U -folded. If there exists a path p in $\Gamma(n-1)$ such that $o(p) = v$, $\overline{\mu(p)} = u_n^k \circ c$, $k \in \mathbb{Z}$ then there exists a path $r = r_1 r_2$ in $\Gamma(n)$ such that $o(r) = o(p)$, $t(r) = t(p)$, $\overline{\mu(r)} = \overline{\mu(p)}$, $r_1 \in \Gamma(n-1)$, $\mu(r_1) = \pi(u_n)^l$, $|k| = |l| + 1$, $t(p_1) \in V(C)$ and r_2 is a (u_n, δ) -irregular path of type I or II, where $\delta k > 0$.*

Proof. Without loss of generality we can assume that $k > 0$ and also that $u_{n-1} = \max\{U(u_n)\}$. Hence

$$u_n = h_1 \circ u_{n-1}^{\alpha_1} \circ h_2 \circ \dots \circ u_{n-1}^{\alpha_m} \circ h_{m+1},$$

where $m \geq 1$ and

$$\pi(u_n) = \pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2) \dots u_{n-1}^{\alpha_m} \pi(h_{m+1}).$$

Since $\pi(u_n \circ u_n) = \pi(u_n) \pi(u_n)$ then

$$\begin{aligned} \pi(u_n^k \circ c) &= (\pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2) \dots u_{n-1}^{\alpha_m} \pi(h_{m+1}))^{k-1} \pi(h_1) u_{n-1}^{\alpha_1} \\ &\quad \times \pi(h_2) \dots \pi(u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c). \end{aligned}$$

Since $\Gamma(n-1)$ is U -folded then by Proposition 4.9 there exists a path r in $\Gamma(n-1)$ such that $o(r) = o(p)$, $t(r) = t(p)$, $\mu(r) = \pi(u_n^k \circ c)$. Hence, there exists an initial subpath r_1 of r such that $t(r_1) \in V(C)$ and

$$\mu(r_1) = (\pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2) \dots u_{n-1}^{\alpha_m} \pi(h_{m+1}))^{k-1},$$

and the initial subpath r_2 of r such that $r = r_1 r_2$ is a (u_n, δ) -irregular path of type I, where $\delta k > 0$. □

Lemma 4.12. *Let Γ be a finite $(\mathbb{Z}[t], X)$ -graph such that $U(\Gamma) = \{u_1, \dots, u_{l(\Gamma)}\}$. Let $u_n \in U(\Gamma)$, C be a u_n -component of Γ , $v \in V(C)$. Suppose*

- (1) $\Gamma(n - 1)$ is U -folded,
- (2) $\Gamma(n)$ satisfies the conditions (1)–(7)
- (3) *there exists a path p in $\Gamma(n)$ such that $o(p) = v$, $\overline{\mu(p)} = u_n^k \circ c$, $k \in \mathbb{Z}$ and c does not have $u_n^{\pm 1}$ as an initial segment.*

Then there exists a path $r = r_1 r_2$ in $\Gamma(n)$ such that $o(r) = o(p)$, $t(r) = t(p)$, $\overline{\mu(r)} = \overline{\mu(p)}$, $r_1 \in \Gamma(n - 1)$ and one of the following holds:

- (1) $\overline{\mu(r_1)} = u_n^k \circ c_1$,
- (2) $\mu(r_1) = \pi(u_n)^l$, $|l| = |k| - 1$, $t(r_1) \in V(C)$ and r_2 is a (u_n, δ) -irregular path of type II, where $\delta k > 0$.

Proof. Without loss of generality we can assume that $k > 0$.

Let p_1 be the maximal initial subpath of p which does not contain edges labeled by u_n^α , $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$. Since $\Gamma(n)$ satisfies the conditions (1)–(7) then we can assume that $p_1 \in \Gamma(n - 1)$. If $\overline{\mu(p_1)} = u_n^k \circ c_1$ then we are done. Suppose on the contrary that $\overline{\mu(p_1)} = u_n^l \circ w_1$ and $l < k$. Hence, $u_n = w_1 \circ w_2$. If $l < k - 1$ then we have a contradiction since we obtain $w_1 \circ w_2 = w_2 \circ w_1$. Thus, $l = k - 1$.

Since $\Gamma(n - 1)$ is U -folded then by Proposition 4.9 there exists a path q in $\Gamma(n - 1)$ such that $o(q) = o(p_1)$, $t(q) = t(p_1)$, $\mu(q) = \pi(u_n^l \circ w_1)$. On the other hand we have

$$\pi(u_n^l \circ w_1) = \pi(u_n^l)\pi(w_1),$$

so q can be subdivided as $q = q_1 q_2$ so that $\mu(q_1) = \pi(u_n^l) = \pi(u_n)^l$, $\mu(q_2) = \pi(w_1)$. Finally, if p_2 is such that $p = p_1 p_2$ then we set $r_1 = q_1$ and $r_2 = q_2 p_2$. Observe that $t(r_1) \in V(C)$ because $\Gamma(n)$ satisfies the conditions (1)–(7) and r_2 is a (u_n, δ) -irregular path of type II, where $\delta = 1$. □

Proposition 4.13. *Let Γ be a finite connected $(\mathbb{Z}[t], X)$ -graph. Then there exists a U -folded $(\mathbb{Z}[t], X)$ -graph Δ which is obtained from Γ by a finite sequence of free and u -foldings. Moreover Δ can be found effectively.*

Proof. To prove the proposition we use the induction on $l(\Gamma)$. If $l(\Gamma) = 0$, that is, all edges in Γ are labeled by letters from X^\pm , then by Lemma 3.6 using finitely many free foldings one can obtain from Γ a partially folded $(\mathbb{Z}[t], X)$ -graph Δ which satisfies all the conditions (1)–(8) since it does not contain u -components.

Assume that the statement of the theorem holds for any graph of level $n - 1$. Let $l(\Gamma) = n$ and let $U(\Gamma) = \{u_1, \dots, u_n\}$.

We construct a $(\mathbb{Z}[t], X)$ -graph Δ from Γ in several steps which are described in the series of claims below.

Claim 4.14. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. By finitely many free and u -foldings, where $u \in U(\Phi)$, one can*

transform Φ into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that $\Phi_0(n - 1)$ is U -folded and all u_n -components of Φ_0 are u_n -folded.

Proof of Claim 4.14. Let C_1, \dots, C_K , $K \in \mathbb{N}$ be the list of all u_n -components of Φ . Using finitely many u_n -foldings every C_i can be transformed into a u_n -folded u_n -component independently of all other C_j , $j \neq i$. Observe that after all these transformations in the resulting graph Φ' we can have that $\Phi'(n - 1)$ is not U -folded any more. But this can happen only if two vertices of $\Phi(n - 1)$ have been identified, that is, if performed u_n -foldings have decreased the number of vertices in the u_n -components and

$$\sum_{i=1}^K |V(C_i)| < \sum_{i=1}^K |V(C'_i)|,$$

where C'_i is a u_n -component of Φ' which is obtained from C_i . By the induction hypothesis we transform Φ' into a new $(\mathbb{Z}[t], X)$ -graph so that $\Phi'(n - 1)$ transforms into $\Phi'(n - 1)$ which is U -folded. Correspondingly, C'_1, \dots, C'_K transform into C''_1, \dots, C''_K and a u_n -component C''_i , $i \in [1, K]$ can be not u_n -folded only if two vertices of C'_i have been identified while transforming Φ' into Φ'' . In this case again we have

$$\sum_{i=1}^K |V(C'_i)| < \sum_{i=1}^K |V(C''_i)|.$$

Both inequalities above show that in a finite number of steps we obtain a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that $\Phi_0(n - 1)$ is U -folded and all u_n -components of Φ_0 are u_n -folded. The claim is proved.

Claim 4.15. Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding finitely many edges to Φ and using finitely many free and u -foldings, where $u \in U(\Phi)$ one can transform Φ into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that

- (a) Φ_0 satisfies (1)–(3),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi, v) = L(\Phi_0, v_0)$.

Proof of Claim 4.15. By Claim 4.14 we can assume $\Phi(n - 1)$ to be U -folded and all u_n -components of Φ to be u_n -folded.

For any edge $e \in E(\Phi)$ such that $\mu(e) = u_n^{k_e}$, $k_e \in \mathbb{Z}$ we add to Φ a new path p_e such that $o(p_e) = o(e)$, $t(p_e) = t(e)$, $\mu(p_e) = \pi(u_n)^{k_e}$. The graph thus obtained we denote by Φ' . Obviously $L(\Phi, v) \subseteq L(\Phi', v)$. On the other hand observe that any reduced path p in Φ' which contains at least one new edge also has a subpath p_e such that there exists $e \in E(\Phi)$ and $o(e) = o(p_e)$, $t(e) = t(p_e)$, $\mu(e) = \overline{\mu(p_e)}$. Thus, substituting p_e by e and continuing by the induction on the number of new edges in p one obtains a path p' in Φ such that $o(p') = o(p)$, $t(p') = t(p)$, $\mu(p') = \overline{\mu(p)}$. Hence, $L(\Phi', v) \subseteq L(\Phi, v)$.

Using the induction hypothesis and Claim 1, we obtain a new graph Φ'' such that $\Phi''(n-1)$ is U -folded and all its u_n -components are u_n -folded. Observe that if there exists $e \in E(\Phi'')$ such that $\mu(e) = u_n^k$, $k \in \mathbb{Z}$ and there exists no path p_e such that $o(p_e) = o(e)$, $t(p_e) = t(e)$, $\mu(p_e) = \pi(u_n)^k$ then it means that transformation of Φ' into Φ'' has affected u_n -components of Φ' , that is,

$$\sum_{i=1}^K |V(C''_i)| < \sum_{i=1}^K |V(C'_i)|,$$

where C'_1, \dots, C'_K , $K \in \mathbb{N}$ is the list of all u_n -components of Φ' and C''_1, \dots, C''_K are the corresponding u_n -components of Φ'' . Hence, continuing the operation described above we can obtain a graph with the required property and in fact we can assume Φ'' to be such a graph. However, u_n -components of Φ'' may be not isolated.

Let Θ be a connected component of $\Phi''(n-1)$ which contains C''_1 and let $z \in V(\Phi''(n-1)) \cap V(C''_1)$. By Lemma 4.10, since Θ is finite we can find all vertices of Θ which are connected with z by paths labeled by finite exponents of u_n . Denote the set of these vertices by $V_{C''_1}$. Suppose there exists $z' \in V_{C''_1} \cap C''_j$, $j \in [1, K]$, $j \neq 1$. Hence C''_1 and C''_j are connected by a path p such that $o(p) = z$, $t(p) = z'$, $\mu(p) = \pi(u_n)^k$, $k \in \mathbb{Z}$. Hence we add to Φ'' an edge e_p labeled by u_n^k and reduce by this the number of u_n -components of Φ'' . Since this number is finite then using the considerations above we obtain a new graph Φ_0 with the properties (1)–(3). The claim is proved.

Claim 4.16. *Let Φ_0 be the $(\mathbb{Z}[t], X)$ -graph obtained in Claim 4.15. Adding finitely many edges to Φ_0 one can transform Φ_0 into a new $(\mathbb{Z}[t], X)$ -graph Φ_1 such that*

- (a) Φ_1 satisfies (1)–(3),
- (b) $\Phi_1(n-1)$ is U -folded,
- (c) if $v_0 \in V(\Phi_0)$ and $v_1 \in V(\Phi_1)$ corresponds to v_0 then $L(\Phi_1, v_1) = L(\Phi_0, v_0)$,
- (d) if C is a u_n -component of Φ_1 , $v \in V(C)$ and there exists a loop p in $\Phi_1(n-1)$ at v such that $\overline{\mu(p)} = u_n^k$, $k \in \mathbb{N}$ then $u_n^k \in H_{u_n}(z_C)$.

Proof of Claim 4.16. Let C_1, \dots, C_K , $K \in \mathbb{N}$ be the list of all u_n -components of Φ_0 . Let Θ be a connected component of $\Phi_0(n-1)$ which contains C_1 and let $z \in V(\Phi_0(n-1)) \cap V(C_1)$. Suppose there exists a loop p in Θ at z such that $\overline{\mu(p)} = u_n^k$, $k > 0$ (observe that we can check this effectively in view of Lemma 4.10). Since $\Phi_0(n-1)$ is U -folded we can assume that $\mu(p) = \pi(u_n)^k$ and k is minimal possible. If $u_n^k \notin H_{u_n}(z)$ then we add to Φ_0 a new edge e such that $o(e) = t(e) = z$, $\mu(e) = u_n^k$. Let Φ'_0 be the resulting graph. Then $V(\Phi'_0) = V(\Phi_0)$ and it is easy to see that for any $v \in V(\Phi_0)$ we have $L(\Phi'_0, v) = L(\Phi_0, v)$. Moreover, Φ'_0 satisfies (1)–(3)

Let

$$M_1(\Phi_0) = \sum_{i=1}^K l_i,$$

where $H_{u_n}(z_{C_i}) \cap \langle u_n \rangle = \langle u_n^{l_i} \rangle$, $l_i > 0$ and $i \in [1, K]$, and

$$M_2(\Phi_0) = |\{C_i \mid H_{u_n}(z_{C_i}) \cap \langle u_n \rangle = \varepsilon, v_i \in V(C_i), i \in [1, K]\}|.$$

The same characteristics can be introduced for Φ'_0 as well and considering ordered pairs $(M_1(\Phi_0), M_2(\Phi_0))$ and $(M_1(\Phi'_0), M_2(\Phi'_0))$ we have that

$$(M_1(\Phi'_0), M_2(\Phi'_0)) < (M_1(\Phi_0), M_2(\Phi_0))$$

with respect to the right lexicographic order. Indeed, if $H_{u_n}(z_{C_1}) \cap \langle u_n \rangle = \varepsilon$ then $M_1(\Phi'_0) < M_1(\Phi_0)$ and $(M_1(\Phi'_0), M_2(\Phi'_0)) < (M_1(\Phi_0), M_2(\Phi_0))$. If $H_{u_n}(z_{C_1}) \cap \langle u_n \rangle = \langle u_n^{l_1} \rangle$, $l_1 > 0$ then $M_1(\Phi'_0) = M_1(\Phi_0)$. Let H be a free abelian group associated with the u_n -component $C_1 \cup \{e\}$ of Φ'_0 . Then $H \cap \langle u_n \rangle = \langle u_n^{l_1} \rangle$ and obviously $l_1 > l'_1$. Hence, $M_2(\Phi'_0) < M_2(\Phi_0)$ and $(M_1(\Phi'_0), M_2(\Phi'_0)) < (M_1(\Phi_0), M_2(\Phi_0))$.

The operation described above can be applied to all u_n -components of Φ_0 and to all their vertices which also belong to $\Phi_0(n - 1)$. The convergence of this process follows from the fact that $M_2(\Psi)$ is bounded for any $(\mathbb{Z}[t], X)$ -graph Ψ and $M_1(\Psi)$ is bounded for any fixed value of $M_2(\Psi)$. After the process stops we obtain the required graph Φ_1 . The claim is proved.

Claim 4.17. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(4),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi_0, v_0) = L(\Phi, v)$.

Proof of Claim 4.17. Applying Claims 4.15 and 4.16 to Φ we obtain a new graph Φ' which has the properties listed in Claim 4.16. It follows from the proofs of the claims above that either $\Phi' = \Phi$ or

$$(M_1(\Phi'), M_2(\Phi'), M_3(\Phi'), M_4(\Phi')) < (M_1(\Phi), M_2(\Phi), M_3(\Phi), M_4(\Phi))$$

with respect to the right lexicographic order, where $M_3(\Psi)$ is the total number of vertices in u_n -components of Ψ and $M_4(\Psi)$ is the number of u_n -components of Ψ .

Let C be a u_n -component of Φ' . If $H_{u_n}(z_C) \cap \langle u_n \rangle = u_n^{k(C)}$, $k(C) > 0$ then for any $v \in V(C)$ we add to Φ' a loop p_v such that $o(p_v) = t(p_v) = v$, $\mu(p_v) = \pi(u_n)^{k(C)}$.

This operation can be performed for all u_n -components of Φ' and let Φ'' be the resulting graph. It is clear that

$$(M_1(\Phi''), M_2(\Phi''), M_3(\Phi''), M_4(\Phi'')) = (M_1(\Phi'), M_2(\Phi'), M_3(\Phi'), M_4(\Phi')).$$

If $\Phi''(n - 1)$ is not U -folded then by iterating application of Claims 4.14–4.16 eventually we obtain the graph Φ_0 with the properties (1)–(4) such that $\Phi_0(n - 1)$ is U -folded. The claim is proved.

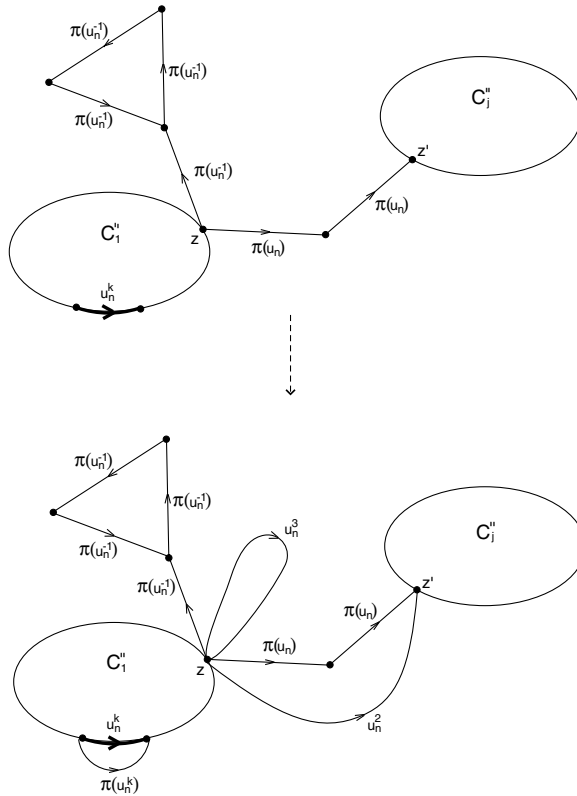


Fig. 4. Getting properties (1)–(4).

Claim 4.18. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding and eliminating finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(5),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi_0, v_0) = L(\Phi, v)$.

Proof of Claim 4.18. In view of Claim 4.17 we can assume that Φ satisfies (1)–(5) and $\Phi(n - 1)$ is U -folded.

Let C be a u_n -component in Φ . Since C is u_n -folded then by definition there exists a positively-oriented path P_C associated with C such that $V(C) = V(P_C)$. Let $v_1, v_2 \in V(C)$, $v_1 \neq v_2$. Consider the following possibilities.

Suppose the subpath $p = e_1 \cdots e_k$ of P_C is such that $o(p) = v_1$, $t(p) = v_2$ contains an edge labeled by an infinite exponent of u_n , say e_j and suppose there exists an integer k_p such that $\overline{\mu(p)} * u_n^{-k_p} \in H_{u_n}(ZC)$. If $H_{u_n}(zC) \cap \langle u_n \rangle = \varepsilon$ then such k_p is unique, while if $H_{u_n}(v_1) \cap \langle u_n \rangle \neq \varepsilon$ then there are infinitely many $r \in \mathbb{Z}$

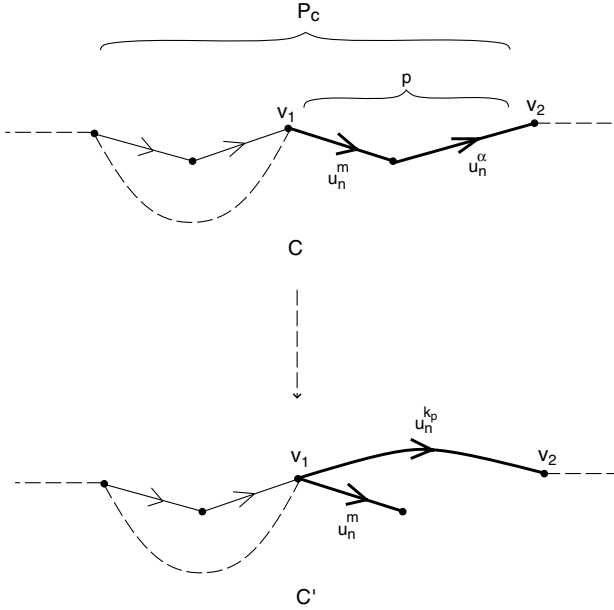


Fig. 5. Getting the property (5) ($\alpha \gg 0$).

such that $\overline{\mu(p)} * u^{-r} \in H_{u_n}(z_C)$ all of which belong to the coset $u^{k_p} * (H_{u_n}(z_C) \cap \langle u_n \rangle)$ and this ensures that it does not matter which one of them is chosen. Thus, we add to C an edge e such that $o(e) = v_1$, $t(e) = v_2$, $\mu(e) = u_n^{k_p}$ but eliminate an edge e_j from P_C . Observe that this operation does not change $H_{u_n}(z_C)$. Moreover, if we denote the resulting graph by Φ' then $V(\Phi') = V(\Phi)$ and for any $v \in V(\Phi)$ we have $L(\Phi', v) = L(\Phi, v)$. In addition, obviously $M_1(\Phi') = M_1(\Phi)$, $M_2(\Phi') = M_2(\Phi)$, $M_3(\Phi') = M_3(\Phi)$, $M_4(\Phi') = M_4(\Phi)$.

Let

$$M_5(\Psi) = \sum_{i=1}^K m(C_i),$$

where C_1, \dots, C_K are u_n -components of Ψ and $m(C_i)$ is the number of edges labeled by infinite exponents of u_n in P_{C_i} , $i \in [1, K]$. Then $(M_1(\Phi'), M_2(\Phi'), M_3(\Phi'), M_4(\Phi'), M_5(\Phi')) < (M_1(\Phi), M_2(\Phi), M_3(\Phi), M_4(\Phi), M_5(\Phi))$ with respect to the right lexicographic order. It follows that using the operation described above one can minimize the number of edges of Φ labeled by infinite exponents of u_n in all u_n -components of Φ . If necessary one can iterate the process together with the claims above to obtain the required graph Φ_0 . The claim is proved.

Claim 4.19. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding and eliminating finitely many edges to Φ one can transform*

it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that

- (a) Φ_0 satisfies (1)–(6),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi_0, v_0) = L(\Phi, v)$.

Proof of Claim 4.19. In view of Claim 4.18 we can assume that Φ satisfies (1)–(5) and $\Phi(n - 1)$ is U -folded.

Let C be a u_n -component in Φ . Since C is u_n -folded then by definition there exists a positively-oriented path P_C associated with C such that $V(C) = V(P_C)$. Let $v_1, v_2 \in V(C)$, $v_1 \neq v_2$.

Suppose there exists a reduced path r in $\Phi(n - 1)$ such that $o(r) = v_1$, $t(r) = v_2$ and $\overline{\mu(r)} = u_n^l$, $l > 0$. Since $\Phi(n - 1)$ is U -reduced by Proposition 4.9 we can assume r to be label reduced and $\mu(r) = \pi(u_n)^l$. Then, there exists a unique label reduced path q in $\Phi(n - 1)$ such that $o(q) = v_1$, $t(q) = v_2$ and $\mu(q) = \pi(u_n)^k$, $l \geq k > 0$ and for any two initial subpaths q_1, q_2 of q such that $\mu(q_1) = \pi(u_n)^{k_1}$, $\mu(q_2) = \pi(u_n)^{k_2}$, $k_1 < k_2 < k$ we have $t(q_1) \neq t(q_2)$.

Since $v_1, v_2 \in V(C)$ then there exists a reduced subpath p of P_C such that $o(p) = v_1$, $t(p) = v_2$, $\overline{\mu(p)} = u_n^\alpha$, $\alpha \in \mathbb{Z}[t]$.

If p consists only of edges labeled by finite exponents of u_n then from the properties (3) and (4) of Φ and from the fact that $\Phi(n - 1)$ is U -folded it follows that there exists a unique label reduced path p' in $\Phi(n - 1)$ such that $o(p') = o(p)$, $t(p') = t(p)$, $\mu(p') = \pi(u_n)^\alpha$. Hence, $o(q) = o(p')$, $t(q) = t(p')$. If p is positive then so is p' and we have $\mu(p') = \mu(q)$, otherwise we have a contradiction either with the choice of q or with the fact that $\Phi(n - 1)$ is U -folded. Thus, by Proposition 4.9 we have $q = p'$. If p is negative then p' is also negative and we have a positively oriented loop in $\Phi(n - 1)$ at v_1 which is a concatenation qp'^{-1} so $\overline{\mu(qp'^{-1})} \in H_{u_n}(z_C)$ otherwise a contradiction with (4).

Thus, we can assume that p contains an edge labeled by an infinite exponent of u_n . Then $\overline{\mu(p)} * \overline{\mu(q)}^{-1} \notin H_{u_n}(z_C)$ otherwise we have a contradiction with (5). So we add to C an edge e labeled by u_n^k such that $o(e) = v_1$, $t(e) = v_2$. Denote the resulting graph by Φ' . Observe that after the performed operation the new u_n -component $C \cup \{e\}$ may not be u_n -folded. Applying Claim 5 to Φ' we obtain a new graph Φ'' and it follows from the proof of Claim 5 that $(M_1(\Phi''), M_2(\Phi''), M_3(\Phi''), M_4(\Phi''), M_5(\Phi'')) < (M_1(\Phi'), M_2(\Phi'), M_3(\Phi'), M_4(\Phi'), M_5(\Phi'))$ with respect to the right lexicographic order. Hence, after finitely many iterations of the process described above we obtain the required graph Φ_0 . The claim is proved.

Claim 4.20. Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding and eliminating finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that

- (a) Φ_0 satisfies (1)–(7),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi_0, v_0) = L(\Phi, v)$.

Proof of Claim 4.20. In view of Claim 4.19 we can assume that Φ satisfies (1)–(6) and $\Phi(n - 1)$ is U -folded.

Let C be a u_n -component in Φ . Using Lemma 4.10 we find all paths labeled by $\pi(u_n)^m$, $m \in \mathbb{Z}$ in $\Phi(n - 1)$ which originate from vertices of C . The endpoints of these paths compose a finite set S .

Let $v_1 \in V(C)$ and suppose there exists a reduced path r in $\Phi(n - 1)$ for which $o(r) = v_1$, $t(r) = v_2$, $v_2 \in S - \notin V(C)$ and $\overline{\mu(r)} = u_n^l$, $l \in \mathbb{Z}$. Since $\Phi(n - 1)$ is U -reduced by Proposition 4.9 we can assume r to be label reduced and $\mu(r) = \pi(u_n)^l$. Also, without loss of generality we can assume $l > 0$. Then, there exists a unique label reduced path p in $\Phi(n - 1)$ for which we have $o(p) = v_1$, $t(p) = v_2$ and $\mu(p) = \pi(u_n)^k$, $l \geq k > 0$ and such that for any two initial subpaths p_1, p_2 of p such that $\mu(p_1) = \pi(u_n)^{m_1}$, $\mu(p_2) = \pi(u_n)^{m_2}$, $m_1 < m_2 < k$ we have $t(p_1) \neq t(p_2)$. Let us denote $a_i = t(p_i)$, where p_i is an initial subpath of p such that $\mu(p_i) = \pi(u_n)^i$, $i \in [1, k]$.

If $u_n^k \in H_{u_n}(z_C)$ then by Proposition 4.9 there exists a unique label reduced cycle q at v_1 in $\Phi(n - 1)$ labeled by the generator of $H_{u_n}(z_C) \cap \langle u_n \rangle$. Hence $p = q$ because of the choice of p and the fact that $\Phi(n - 1)$ is U -folded. So $v_1 = v_2$ — contradiction. Hence $u_n^k \notin H_{u_n}(z_C)$.

Let v be any vertex in $V(C)$ and let $q_v = e_1 \cdots e_m$ be a label reduced subpath of P_C such that $o(q_v) = v$, $t(q_v) = v_1$, $\mu(e_i) = u_n^{\alpha_i}$, $i \in [1, m]$ (q_v exists since C is u_n -reduced).

(i) Suppose there exists $v \in V(C)$ such that $\overline{\mu(q_v)} * u_n^k \in H_{u_n}(z_C)$. Then all edges in q_v are labeled by finite exponents of u_n — otherwise (5) breaks for Φ and we have a contradiction. Finally, since Φ has properties (3) and (4) and $\Phi(n - 1)$ is U -folded it means that p connects two vertices in C and, hence, $v_2 \in C$ — contradiction.

(ii) Suppose for any $v \in V(C)$ we have $\overline{\mu(q_v)} * u_n^k \notin H_{u_n}(z_C)$. Then we add to C a path $f_1 f_2 \cdots f_k$ so that $o(f_1) = v_1$, $t(f_i) = o(f_{i+1}) = a_i$, $i \in [1, k - 1]$, $t(f_k) = v_2$, $\mu(f_i) = u_n$, $i \in [1, k]$. Applying k u_n -foldings we fold $f_1 f_2 \cdots f_k$ with P_C and the obtained u_n -component is u_n -folded because of our assumption on k . If we denote resulting graph by Φ' then $\Phi'(n - 1)$ is U -folded because v_2 has not been identified with any vertex of Φ and Φ' has the properties (1)–(6). Also we have $M_5(\Phi') = M_5(\Phi)$, although $M_3(\Phi') = M_3(\Phi) + k$.

Since S is finite then applying the operation described above we add to C all vertices from S and the same can be done for all u_n -components of Φ . On each step the resulting graph has the properties (1)–(6) and its $n - 1$ -level is U -folded. In finitely many steps we get a the required graph Φ_0 . The claim is proved.

Claim 4.21. Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n - 1)$ is U -folded and Φ satisfies (1)–(7). Let C be a u_n -component of Φ , $v \in V(C)$ and let p be a (u_n, δ) -irregular path p of type I originating from v . Then either p is doubled in $\Phi(n - 1)$ or there exists a path $r = r_1 r_2$ in $\Phi(n - 1)$ such that $o(r) = o(p)$, $t(r) = t(p)$, $\overline{\mu(r)} = \overline{\mu(p)}$,

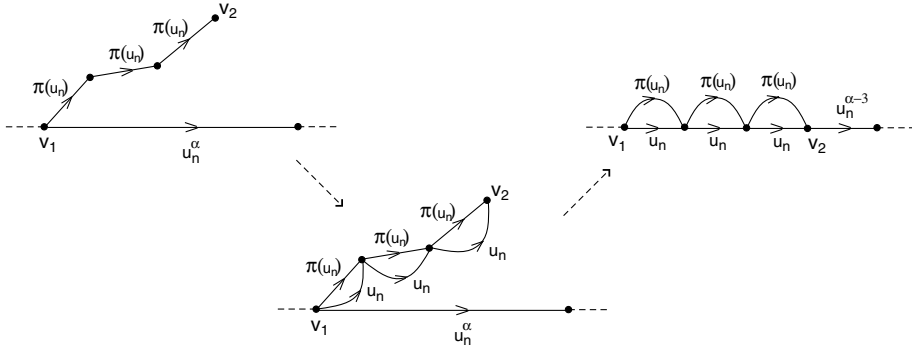


Fig. 6. Getting the property (7).

$\mu(r_1) = \pi(h_1)u_{n-1}^{\alpha_1}\pi(h_2)\cdots\pi(h_m)$, $\overline{\mu(r_2)} = u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c$, $u_{n-1}^\gamma = h_{m+1} \circ u'$, where $\gamma \in \{1, -1\}$, $\alpha_m \gamma > 0$ and

$$\pi(u_n)^\delta = \pi(h_1)u_{n-1}^{\alpha_1}\pi(h_2)\cdots u_{n-1}^{\alpha_m}\pi(h_{m+1}).$$

Proof of Claim 4.21. Without loss of generality we can assume $\delta = 1$. Let

$$p = d_1p_1d_2\cdots d_kp_kd_{k+1},$$

where p_i is a path in some u_{n-1} -component of $\Phi(n-1)$ and d_i is a path in $\Phi(n-1)$ which does not contain edges labeled by u_{n-1}^α , $\alpha \in \mathbb{Z}[t]$. We can assume that there exists no initial subpath d'_i of $d_i p_i \cdots d_k p_k d_{k+1}$ such that $\overline{\mu(d'_i)} = u_{n-1}^{\pm 1}$ and there exists no terminal subpath d''_i of d_i such that $\overline{\mu(d''_i)} = u_{n-1}^{\pm 1}$ — otherwise, since $\Phi(n-1)$ is U -folded, by induction we can assume that $t(d'_i)$ and $o(d''_i)$ belong to some u_{n-1} -components and there exist edges e', e'' such that $o(e') = o(d'_i)$, $t(e') = t(d'_i)$, $\mu(e') = u_{n-1}^{\pm 1}$, $o(e'') = o(d''_i)$, $t(e'') = t(d''_i)$, $\mu(e'') = u_{n-1}^{\pm 1}$, so that we can substitute d'_i, d''_i by e', e'' correspondingly and consider the path p' obtained from p by such substitutions. Let $\overline{\mu(p_i)} = u_{n-1}^{\beta_i}$.

Observe that by the properties of standard decompositions h_1 does not have u_{n-1} as a terminal segment. On the other hand $|\overline{\mu(d_1)}| \geq |h_1|$ because of the choice of h_1 .

(i) If $|\overline{\mu(d_1)}| = |h_1|$ then automatically $\overline{\mu(d_1)} = h_1$. Now, either $\beta_1 \leq \alpha_1$ or $\beta_1 > \alpha_1$, $u_{n-1}^\gamma = h_2 \circ u'$, $\gamma \in \{1, -1\}$, $u' \neq \varepsilon$ and $\pi(u_n) = \pi(h_1)u_{n-1}^{\alpha_1}\pi(h_2)$ — otherwise a contradiction with the choice of h_2 .

In the former case, if $\beta_1 = \alpha_1$ then we proceed with d_2 , as before with d_1 , but if $\beta_1 < \alpha_1$, then $\overline{\mu(d_2)}$ contains u_{n-1} as an initial segment, that is, d_2 is $(u_{n-1}, 1)$ -irregular of type I or II and by induction there exists a doubling path $z = z_1 z_2$ for d_2 such that $o(d_2) = o(z)$, $t(d_2) = t(z)$, $\overline{\mu(z)} = \overline{\mu(p)}$, $\mu(z_1) = \pi(u_{n-1})$, $t(z_1) = o(z_2)$ and $t(p_1)$ belong to the same u_{n-1} -component D . So, we have either a path $d_1 p_1$ or a path $d_1 p_1 z_1$ labeled by $\pi(h_1)u_{n-1}^{\alpha_1}$.

In the latter case, $u_n = h_1 \circ u_{n-1}^{\alpha_1} \circ h_2$ and there exists the required path $r = r_1 r_2$, where $\mu(r_1) = \pi(h_1)$ and $\overline{\mu(r_2)} = u_{n-1}^{\alpha_1} \circ h_2 \circ c$, $u_{n-1}^\gamma = h_2 \circ u'$, $\gamma \in \{1, -1\}$.

(ii) If $|\overline{\mu(d_1)}| > |h_1|$ then by [10, Lemma 6.9] we should have $\overline{\mu(d_1)} = h_1 \circ u_{n-1}^k$, $k \in \mathbb{Z}$, where the sign of k depends on the sign of α_1 and we can assume $k > 0$. In fact, because of our assumption about d_i we have $k = 1$. Hence, d_1 is $(u_{n-1}, 1)$ -irregular of type I or II and by induction there exists a doubling path $z = z_1 z_2$ for d_1 such that $o(d_1) = o(z)$, $t(d_1) = t(z)$, $\mu(z_2) = \pi(u_{n-1})$, $t(z_1) = o(z_2)$ belongs to some u_{n-1} -component D . Observe that $t(z_1)$ and $t(d_1)$ belong to the same u_{n-1} -component D .

We proceed with p_1 . We have either $\beta_1 + 1 \leq \alpha_1$ or $\beta_1 + 1 > \alpha_1$, $u_{n-1}^\gamma = h_2 \circ u'$, $\gamma \in \{1, -1\}$, $u' \neq \varepsilon$ and $\pi(u_n) = \pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2)$ — otherwise a contradiction with the choice of h_2 .

If $\beta_1 + 1 = \alpha_1$ then we proceed with d_2 , as before with d_1 . If $\beta_1 + 1 < \alpha_1$, then $\overline{\mu(d_2)}$ contains u_{n-1} as an initial segment, that is, d_2 is $(u_{n-1}, 1)$ -irregular of type I or II and by induction there exists a doubling path $z' = z'_1 z'_2$ for d_2 such that $o(d_2) = o(z')$, $t(d_2) = t(z')$, $\mu(z'_1) = \pi(u_{n-1})$ and $t(z'_1) = o(z'_2)$, $t(p_1)$ belong to the same u_{n-1} -component D . So, we have either a path $z_1 z_2 p_1$ or a path $z_1 z_2 p_1 z'_1$ labeled by $\pi(h_1) u_{n-1}^{\alpha_1}$.

If $\beta_1 + 1 > \alpha_1$, $u_{n-1}^\gamma = h_2 \circ u'$, $\gamma \in \{1, -1\}$, $u' \neq \varepsilon$ and $\pi(u_n) = \pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2)$ then $u_n = h_1 \circ u_{n-1}^{\alpha_1} \circ h_2$ and there exists the required path $r = r_1 r_2$, where $\mu(r_1) = \pi(h_1)$ and $\overline{\mu(r_2)} = u_{n-1}^{\alpha_1} \circ h_2 \circ c$.

We have considered the first step and now the proof of the claim follows by induction on m .

Claim 4.22. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n-1)$ is U -folded and Φ satisfies (1)–(7). If C is a u_n -component of Φ , $v \in V(C)$ and there is a doubled (u_n, δ) -irregular path p of type I originating from v then any (u_n, δ) -irregular path r of type I originating from v is also doubled.*

Proof of Claim 4.22. Suppose there exists an undoubled (u_n, δ) -irregular path r of type I originating from v . By Claim 4.21 we have $r = z_1 z_2$, $\mu(z_1) = \pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2) \cdots \pi(h_m)$, $\overline{\mu(z_2)} = u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c(r)$, $u_{n-1}^\gamma = h_{m+1} \circ u'$, where $\gamma \in \{1, -1\}$ and

$$\pi(u_n)^\delta = \pi(h_1) u_{n-1}^{\alpha_1} \pi(h_2) \cdots u_{n-1}^{\alpha_m} \pi(h_{m+1}).$$

Since p is doubled then there exists a label reduced path $q = q_1 q_2$ in $\Phi(n-1)$ such that $o(q) = v$, $t(q) = t(q_2) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)}$, $\mu(q_1) = \pi(u_n)^\delta$, $t(q_1) \in V(C)$. Observe that $q_1 = z_1 r_1$, where $\mu(r_1) = u_{n-1}^{\alpha_m} \pi(h_{m+1})$. We use the notation $r_1 = r_2 r_3$, where $\mu(r_2) = u_{n-1}^{\alpha_m}$ and $\mu(r_3) = \pi(h_{m+1})$.

Consider $a(r) = u_{n-1}^{\alpha_m} \circ h_{m+1} \circ c(r)$. Without loss of generality we can assume $\alpha_m > 0$, so $\gamma = 1$. Since $\alpha_m \gg 0$ it follows that

$$\pi(a(r)) = u_{n-1}^{\beta(r)} \circ a_1 \circ \cdots \circ a_k$$

and $\beta(r) \geq \alpha_m + \gamma$. $\Phi(n-1)$ is U -folded so there exists a path d in $\Phi(n-1)$ such that $o(d) = o(z_2)$, $t(d) = t(z_2)$, $\mu(d) = \pi(a(r))$ and we denote by d_1 the initial subpath of d such that $\mu(d_1) = u_{n-1}^{\beta(r)}$. Observe that $t(r_2) \sim_{u_{n-1}} t(d_1)$. Since $\Phi(n-1)$ is U -folded there exists a continuation r'_3 of r_3 such that $t(r'_3) \sim_{u_{n-1}} o(r_3) \sim_{u_{n-1}} t(d_1)$. Hence, there is a label reduced path z such that $o(z) = t(r_3)$, $t(z) = t(r)$,

$$\overline{\mu(z)} = u' \circ u_{n-1}^{\beta(r) - \alpha_m - \gamma} \circ a_1 \circ \cdots \circ a_k$$

and $q_1 z$ is label reduced. Finally, $\overline{\mu(q_1 z)} = \overline{\mu(r)}$, $o(q_1 z) = o(r)$, $t(q_1 z) = t(r)$ so $q_1 z$ is a doubling of r — contradiction with our assumption. The claim is proved.

Claim 4.23. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n-1)$ is U -folded and Φ satisfies (1)–(7). If C is a u_n -component of Φ , $v \in V(C)$ and there are (u_n, δ) -irregular paths $p = \overline{p_{\Phi(n-1)} p_{\Phi(n)}}$, $r = \overline{r_{\Phi(n-1)} r_{\Phi(n)}}$ of type II originating from v then $\overline{\mu(p_{\Phi(n-1)})} = \overline{\mu(r_{\Phi(n-1)})}$.*

Proof of Claim 4.23. Without loss of generality we can assume $\delta = 1$. We have $p = p_{\Phi(n-1)} p_{\Phi(n)}$, $p_{\Phi(n-1)} \in \Phi(n-1)$, $\overline{\mu(p_{\Phi(n-1)})} = w_1$, $\overline{\mu(p_{\Phi(n)})} = w_2 \circ c(p) = u_n^{\gamma_p} \circ c_1(p)$, $\gamma_p \in \mathbb{Z}[t] - \mathbb{Z}$, $u_n = w_1 \circ w_2$, $w_2 \neq \varepsilon$ and $r = r_{\Phi(n-1)} r_{\Phi(n)}$, $r_{\Phi(n-1)} \in \Phi(n-1)$, $\overline{\mu(r_{\Phi(n-1)})} = w_3$, $\overline{\mu(r_{\Phi(n)})} = w_4 \circ c(r) = u_n^{\gamma_r} \circ c_1(r)$, $\gamma_r \in \mathbb{Z}[t] - \mathbb{Z}$, $u_n = w_3 \circ w_4$, $w_4 \neq \varepsilon$.

Observe that if $\gamma_p \gamma_r > 0$, that is, for example $\gamma_p > 0$, $\gamma_r < 0$ then we have $w_3 \circ w_4 = w_3 \circ w_4^{-1}$ and $w_4 = w_4^{-1} = \varepsilon$ — contradiction. So, we can assume $\gamma_p > 0$, $\gamma_r > 0$.

Suppose $|w_1| > |w_3|$. Then $w_1 = w_3 \circ a$, $a \neq \varepsilon$. Observe that $w_1 \circ u_n^{\gamma_p}$ and $w_3 \circ u_n^{\gamma_r}$ have common initial segment of the length at least $2|u_n|$ and we obtain

$$w_1 \circ w_3 \circ w_4 \circ w_3 = w_3 \circ w_1 \circ w_2 \circ w_1.$$

It follows that $[w_1, w_3] = [w_1, a] = [w_3, a] = \varepsilon$. On the other hand we have

$$w_4 \circ w_3 = w_2 \circ w_1.$$

Since $w_4 = a \circ w_2$ then $a \circ w_2 \circ w_3 = w_2 \circ w_3 \circ a$ and using the fact that $w_3 \circ a = a \circ w_3$ we obtain $w_2 \circ a = a \circ w_2$. Hence, $[w_2, a] = [w_2, w_1] = \varepsilon$. It follows that w_1, w_2 are powers of the same element and u_n is a power — contradiction.

Hence, $|w_1| = |w_3|$ and thus $w_1 = w_3$, $w_2 = w_4$. The claim is proved.

Claim 4.24. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $U(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n-1)$ is U -folded and Φ satisfies (1)–(7). If $v \in V(\Phi)$ and p a path in $\Phi(n-1)$ such that $o(p) = v$, $\mu(p) = \pi(w)$ then for any w_1, w_2 such that $w = w_1 \circ w_2$ one can construct a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(7),
- (b) $\Phi(n-1)$ is U -folded,
- (c) if $v_1 \in V(\Phi)$ and $v_2 \in V(\Phi_0)$ corresponds to v_1 then $L(\Phi, v_1) = L(\Phi_0, v_2)$,

(d) *there exists a path $q = q_1q_2$ in Φ_0 such that $o(q) = v$, $t(q) = t(p)$, $\mu(q_1) = \pi(w_1)$, $\mu(q_2) = \pi(w_2)$.*

Proof of Claim 4.24. Let

$$\pi(w) = \pi(h_1)u_{n-1}^{\alpha_1}\pi(h_2) \cdots u_{n-1}^{\alpha_m}\pi(h_{m+1}).$$

We conduct the proof by induction on $k = \max\{U(w_1)\}$.

Let $k = 0$. Hence $w \in F(X)$. Let r_1r_2 be the initial subpath of p such that $r_2 \in E(\Phi)$, $\mu(r_1) = \pi(g)$, $g \in F(X)$, $\mu(r_2) = u_1^\alpha$, $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$. We can assume that $u_1 \in F(X)$ and $\alpha > 0$.

If $|w_1| \leq |g|$ then Φ already has the required path and we are done. Hence assume that $|w_1| > |g|$. It follows that $w_1 = g \circ u_1^k \circ c$, where c is an initial subpath of u_1 (possibly trivial) and $k \geq 0$.

Observe that $o(r_2)$ belongs to some u_1 -component of Φ . If $H_{u_1}(z_D) \cap \langle u_1 \rangle \neq \varepsilon$ then we are done setting $\Phi_0 = \Phi$. Thus, assume $H_{u_1}(z_D) \cap \langle u_1 \rangle = \varepsilon$. If there exists a path in D originating from v and labeled by u_1^{k+1} then there exists a corresponding path in $\Gamma(0)$ with the same label and endpoints — we are done in this case. So, we can assume that there is no such path in D . We split r_2 into $f_1f_2 \cdots f_{k+2}$ so that $\mu(f_i) = u_1$, $i \in [1, k + 1]$, $\mu(f_{k+2}) = u_1^{\alpha-1}$ and add paths z_i , $i \in [1, k + 1]$ so that $o(z_i) = o(f_i)$, $t(z_i) = t(f_i)$, $\mu(z_i) = \pi(u_1)$. Thus obtained graph Φ_0 is U -folded and there exists a path $q = q_1q_2$ in Φ_0 such that $o(q) = v$, $t(q) = t(p)$, $\mu(q_1) = \pi(w_1)$, $\mu(q_2) = \pi(w_2)$.

We assume that we can construct the required graph Φ_0 for any w_1 such that $k = \max\{U(w_1)\} = n - 2$. Suppose $k = n - 1$.

Let r_1r_2 be the initial subpath of p such that $\mu(r_1) = \pi(h_1)$, $\overline{\mu(r_2)} = u_{n-1}^{\alpha_1}$. We can assume that $\alpha_1 > 0$.

Since $\max\{U(w_1)\} = n - 1$ it follows that $w_1 = h_1 \circ u_{n-1}^\theta \circ c$, $\alpha_1 \geq \theta \gg 0$, $u_{n-1} = c \circ c_1$. Observe that $o(r_2)$ belongs to some u_{n-1} -component D and we can assume that there exists no path in D originating from $o(r_2)$ and labeled by u_{n-1}^θ .

We add an edge e_1 and a vertex v_1 to Φ so that $o(e_1) = o(r_2)$, $t(e_1) = v_1$, $\mu(e_1) = u_{n-1}^\theta$. Thus we have a u_{n-1} -component $D \cup \{e_1\}$ in the resulting graph. We fold e_1 with P_D and obtain a new graph Φ' with the u_{n-1} -component D' which corresponds to $D \cup \{e_1\}$. It is easy to see that $H_{u_{n-1}}(z_{D'}) = H_{u_{n-1}}(z_D)$. By our assumption v_1 is not identified with any vertex of D in D' hence v_1 divides some edge $f \in E(D)$ into two edges f_1 and f_2 so that $o(f_1) = o(f)$, $t(f_1) = o(f_2)$, $t(f_2) = t(f)$, $\mu(f) = u_{n-1}^\lambda$, $\mu(f_1) = u_{n-1}^{\lambda_1}$, $\mu(f_2) = u_{n-1}^{\lambda_2}$. Observe that $|\lambda| \gg 0$ and we can assume λ to be positive.

1. $\lambda_1, \lambda_2 \gg 0$.

We can assume that D' has the property (5), otherwise the situation reduces to the case when either λ_1 or λ_2 is finite. Hence, there exists no path in D' originating from v_1 and labeled by u_{n-1} .

(a) $H_{u_{n-1}}(z_{D'}) \cap \langle u_{n-1} \rangle = u_{n-1}^k \neq \varepsilon$.

Let ϕ be a graph which consists of a single loop labeled by $\pi(u_{n-1})^k$. Since $l(\phi) = n - 2$ then by the induction hypothesis there exists a corresponding U -folded graph ϕ' . Let $a_i, i \in [0, k - 1]$ be the vertices of ψ which correspond to the endpoints of paths in ψ starting at $o(\phi)$ and labeled by $\pi(u_{n-1})^i, i \in [0, k - 1]$, and let $\theta = \theta_1\theta_2 \cdots \theta_k$ be a graph which consists of a single loop such that $t(\theta_i) = o(\theta_{i+1}), i \in [1, k - 1], t(\theta_k) = o(\theta_1), \mu(\theta_i) = u_{n-1}, i \in [1, k]$. Then we identify a_i with $o(\theta_{i+1})$ for all $i \in [0, k - 1]$. It easy to see that $\phi' \cup \theta$ is U -folded and we denote it by Ψ .

Now we add Ψ to Φ' so that a_0 is identified with v_1 and fold θ with $P_{D'}$ (in fact with f_2). Hence we obtain a new graph Φ'' with the u_{n-1} -component D'' which corresponds to D' and such that $\Phi''(n - 1)$ is U -folded. Observe that there exists an edge e_2 in D'' such that $o(e_2) = v_1, \mu(e_2) = u_{n-1}$ and a path d in $\Phi''(n - 2)$ such that $o(d) = v_1, t(d) = t(e_2), \mu(d) = \pi(u_{n-1})$.

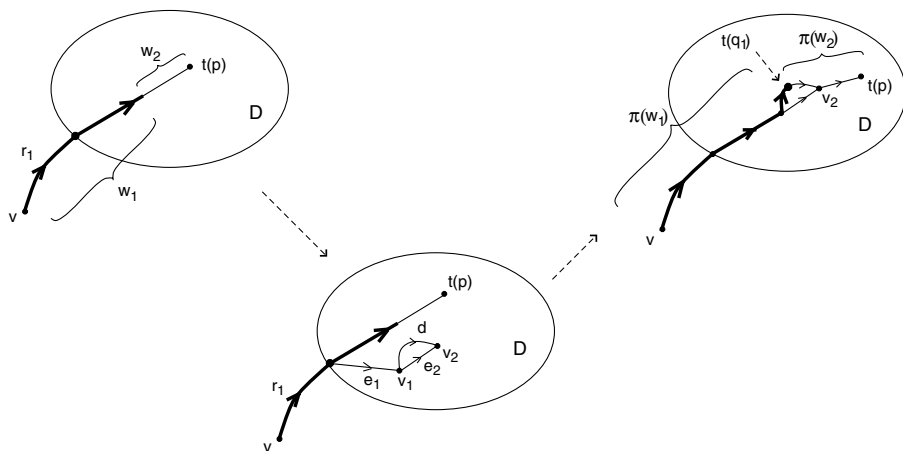


Fig. 7. Proof of Claim 4.24.

(b) $H_{u_{n-1}}(z_{D'}) \cap \langle u_{n-1} \rangle = \varepsilon$.

We divide f_2 into two edges e_2 and f'_2 such that $o(e_2) = o(f_2), t(e_2) = o(f'_2), t(f'_2) = t(f_2), \mu(e_2) = u_{n-1}, \mu(f'_2) = u_{n-1}^{\lambda_2-1}$. Let d be a graph which consists of a single path labeled by $\pi(u_{n-1})$. Since $l(d) = n - 2$ then by the induction hypothesis there exists a corresponding U -folded graph ψ . Abusing notation, we denote the vertices corresponding to endpoints of d in ψ by $o(d)$ and $t(d)$. Finally we add a path ψ to Φ' so that $o(d) = o(e_2), t(d) = t(e_2)$ and obtain a new graph Φ'' with the u_{n-1} -component D'' which corresponds to D' so that $\Phi''(n - 1)$ is U -folded.

2. Either λ_1 or λ_2 is finite.

We can assume that $\lambda_1 \in \mathbb{Z}$. Hence $H_{u_{n-1}}(z_{D'}) \cap \langle u_{n-1} \rangle = \varepsilon$ and then there exists no path in D' originating from v_1 and labeled by u_{n-1} . Then we divide f_2

into two edges e_2 and f'_2 such that $o(e_2) = o(f_2)$, $t(e_2) = o(f'_2)$, $t(f'_2) = t(f_2)$, $\mu(e_2) = u_{n-1}$, $\mu(f'_2) = u_{n-1}^{\lambda_2-1}$. Similarly to 1(b) we add to $\Phi'(n-2)$ a U -folded graph corresponding to a single path ϕ_1 labeled by $\pi(u_{n-1})^{\lambda_1}$ so that $o(\phi_1)$ is identified with $o(f_1)$ and $t(\phi_1)$ is identified with $t(f_1)$, and in the same way we add a U -folded graph corresponding to a single path ϕ_2 labeled by $\pi(u_{n-1})$ so that $o(\phi_2)$ is identified with $o(e_2)$ and $t(\phi_2)$ is identified with $t(e_2)$. The obtained graph with the u_{n-1} -component D'' which corresponds to D' we denote by Φ'' , and it is easy to see that $\Phi''(n-1)$ is U -folded.

Eventually, in all cases above in the graph Φ'' there exists a path p_1dp_2 such that $o(p_1) = v$, $t(p_1) = o(d) = v_1$, $t(d) = o(p_2)$, $t(p_2) = t(p)$, $\mu(p_1) = \pi(h_1)u_{n-1}^\theta$, $\mu(d) = \pi(u_{n-1})$, $\mu(p_2) = u_{n-1}^{\alpha_1-\theta}\pi(h_2)\cdots u_{n-1}^{\alpha_m}\pi(h_{m+1})$. By the induction hypothesis considering d we can construct a graph Φ_0 in which there exists a path d_1d_2 such that $o(d_1) = o(d)$, $t(d_1) = o(d_2)$, $t(d_2) = t(d)$, $\mu(d_1) = \pi(c)$, $\mu(d_2) = \pi(c_1)$. Hence, setting $q_1 = p_1d_1$, $q_2 = d_2p_2$ we get the required.

Next, in the same way we consider $w_1 = h_1 \circ u_{n-1}^{\alpha_1} \circ h'_2(r)$, where h'_2 is an initial segment of h_2 and so on. In a finite number of steps we obtain the required graph Φ_0 . The claim is proved.

Claim 4.25. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n-1)$ is U -folded and Φ satisfies (1)–(7). If C is a u_n -component of Φ , $v \in V(C)$ and there is a (u_n, δ) -irregular path p of type I or II originating from v then adding finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(8),
- (b) $\Phi_0(n-1)$ is U -folded,
- (c) p is doubled in Φ_0 ,
- (d) if $w \in V(\Phi)$ and $w_0 \in V(\Phi_0)$ corresponds to w then $L(\Phi_0, w_0) = L(\Phi, w)$.

Proof of Claim 4.25. Without loss of generality we can assume $\delta = 1$. We have the following cases.

I. p is of type I.

Since u_n^δ is an initial subword of $w = \overline{\mu(p)}$ then by Claim 4.24 we transform Φ into the graph Φ' such that there exists a path $q = q_1q_2$ in $\Phi'(n-1)$ and $o(q) = v$, $t(q) = t(q_2) = t(p)$, $\overline{\mu(q)} = \overline{\mu(p)}$, $\mu(q_1) = \pi(u_n)^\delta$, $\mu(q_2) = \pi(u_n^{-\delta} * w)$. Finally to make this path q a doubling of p we need just to attach $t(q_1)$ to C .

II. p is of type II.

We have $p = p_{\Phi(n-1)} p_{\Phi(n)}$, $o(p) \in V(C)$, $p_{\Phi(n-1)} \in \Phi(n-1)$, $\overline{\mu(p_{\Phi(n-1)})} = w_1$, $\overline{\mu(p_{\Phi(n)})} = w_2 \circ c = u_n^\gamma \circ c_1$, $\gamma \in \mathbb{Z}[t] - \mathbb{Z}$, $u_n = w_1 \circ w_2$, $w_2 \neq \varepsilon$, $\delta \in \{1, -1\}$. That is, $t(p_{\Phi(n-1)}) \in V(D)$, where D is a u_n -component of $\Phi(n)$. Without loss of generality we can assume $\gamma > 0$. Since $\gamma \gg 0$ it follows that $\pi(u_n^\gamma \circ c_1) = u_{n-1}^{\gamma'} \circ \pi(c_2)$, where $\gamma' \geq \gamma$ and c_2 is a terminal segment of c_1 .

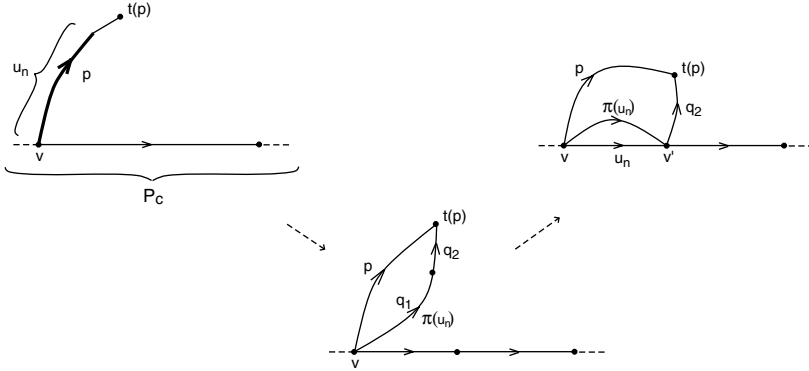


Fig. 8. Doubling of a $(u_n, 1)$ -irregular path of type I. Possible case 1.

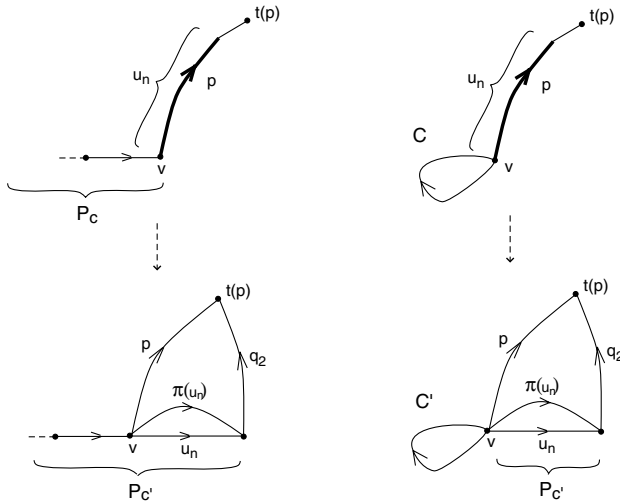


Fig. 9. Doubling of a $(u_n, 1)$ -irregular path of type I. Possible cases 2 and 3.

1. The first edge of $p_{\Phi(n)}$ is labeled by finite exponent of u_n .

Then there exists a label reduced path d labeled by $\pi(u_n)$ such that $o(d) = t(p_{\Phi(n-1)})$, $t(d) \in V(D)$. On the other hand there exists a path p' such that $o(p') = t(d)$, $t(p') = t(p)$, $\mu(p') = c_2$, thus $p_{\Phi(n-1)}dp'$ has the same label and endpoints as p . Now observe that $p_{\Phi(n-1)}d$ is $(u_n, 1)$ -irregular of type I and the existence of its doubling implies the existence of a doubling for p . Hence, we reduce everything to the case I.

2. The first edge e of $p_{\Phi(n)}$ is such that $o(e) = o(p_{\Phi(n)})$, $\mu(e) = u_n^\beta$, $\beta \gg 0$.

(a) $H_{u_n}(z_D) \cap \langle u_n \rangle \neq \varepsilon$.

By the properties (3) and (4) of Φ there exists a label reduced path d labeled by $\pi(u_n)$ such that $o(d) = t(p_{\Phi(n-1)})$, $t(d) \in V(D)$. Also there exists a path p' such

that $o(p') = t(d)$, $t(p') = t(p)$, $\overline{\mu(p')} = c_2$, thus $p_{\Phi(n-1)}dp'$ has the same label and endpoints as p . Hence we have the reduction to the case I as in 1.

(b) $H_{u_n}(z_D) \cap \langle u_n \rangle = \varepsilon$.

Observe that there exists no path in D starting at $t(p_{\Phi(n-1)})$ with the reduced label u_{n-1} — otherwise D is not minimal with respect to the number of infinite edges and we get a contradiction with the fact that Φ has the properties (1)–(7).

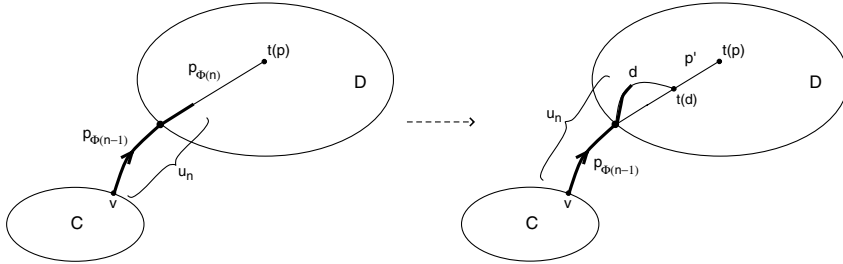


Fig. 10. Doubling of a $(u_n, 1)$ -irregular path of type II. Reduction to I.

We divide e by a new vertex v' into two edges e_1 and e_2 so that $o(e_1) = o(e)$, $t(e_1) = v'$, $o(e_2) = v'$, $t(e_2) = t(e)$, $\mu(e_1) = u_n$, $\mu(e_2) = u_n^{\beta-1}$. The resulting graph we denote by Φ' and the u_n -component corresponding to D by D' . Observe that D' is u_n -folded because of our assumption. Finally, we add a U -folded graph corresponding to a single path d labeled by $\pi(u_n)$ so that $o(d)$ is identified with $o(e_1)$ and $t(d)$ is identified with $t(e_1)$. The resulting graph Φ'' has the properties (1)–(7) and $\Phi''(n - 1)$ is U -folded. Finally, since $t(d) \in V(D')$ then there exists a path p' such that $o(p') = t(d)$, $t(p') = t(p)$, $\overline{\mu(p')} = c_2$, thus $p_{\Phi(n-1)}dp'$ has the same label and endpoints as p . Everything reduces to the case I again.

This completes the proof of the claim.

Claim 4.26. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Suppose $\Phi(n - 1)$ is U -folded and Φ satisfies (1)–(7). If C is a u_n -component of Φ , $v \in V(C)$ then adding finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(8),
- (b) $\Phi_0(n - 1)$ is U -folded,
- (c) all (u_n, δ) -irregular paths of type III originating from v are doubled in Φ_0 ,
- (d) if $w \in V(\Phi)$ and $w_0 \in V(\Phi_0)$ corresponds to w then $L(\Phi_0, w_0) = L(\Phi, w)$.

Proof of Claim 4.26. If there is no (u_n, δ) -irregular path of type III originating from v then we are done setting $\Phi_0 = \Phi$.

Suppose on the contrary that there exists a path p in Φ such that $o(p) = v$, $\overline{\mu(p)} = w_1(p)$, $u_n^\delta = w_1(p) \circ w_2(p)$, $w_2(p) \neq \varepsilon$, $\delta \in \{1, -1\}$ and there exists a path p' in C such that $o(p') = o(p)$, $\overline{\mu(p')} = u_n^\gamma$, $\gamma \in \mathbb{Z}[t]$, $\gamma\delta > 0$.

Without loss of generality we can assume $\delta = 1$, hence $\gamma > 0$.

Observe that if $\gamma \in \mathbb{Z}$ then the first edge of p' is labeled by a finite exponent of u_n because of the properties (1)–(8) of Φ . In this case it follows immediately that all $(u_n, 1)$ -irregular paths of type III originating from v are doubled. This happens also when $H_{u_n}(z_C) \cap \langle u_n \rangle \neq \varepsilon$ or there exists an edge f in C such that $o(f) = v$ and $\mu(f) = u_n^k$, $k \in \mathbb{Z}$. Hence we can assume that $H_{u_n}(z_C) \cap \langle u_n \rangle = \varepsilon$ and that the first edge e of p' belongs to P_C and is labeled by an infinite positive exponent of u_n , that is, $o(e) = v$, $\mu(e) = u^\beta$, $\beta \gg 0$.

Observe that there exists no path in C starting at $t(p_{\Phi(n-1)})$ with the reduced label u_{n-1} — otherwise C is not minimal with respect to the number of infinite edges and we get a contradiction with the fact that Φ has the properties (1)–(7).

We divide e by a new vertex v' into two edges e_1 and e_2 so that $o(e_1) = o(e)$, $t(e_1) = v'$, $o(e_2) = v'$, $t(e_2) = t(e)$, $\mu(e_1) = u_n$, $\mu(e_2) = u_n^{\beta-1}$. The resulting graph we denote Φ' and the resulting u_n -component by C' . Hence C' is u_n -folded.

Let d be a graph which consists of a single path labeled by $\pi(u_n)$. Since $l(d) = n - 1$ then by the induction hypothesis there exists a corresponding U -folded graph ψ . Abusing notation we denote the vertices corresponding to endpoints of d in ψ by $o(d)$ and $t(d)$. Finally we add a path ψ to Φ' so that $o(d) = o(e_1)$, $t(d) = t(e_1)$ and obtain a new graph Φ'' with the u_{n-1} -component C'' which corresponds to C' .

Observe that $\Phi''(n-1)$ is not U -folded since there is at least one $(u_n, 1)$ -irregular path p of type III originating from $v = o(d)$.

Let \mathcal{P} be the set of all $(u_n, 1)$ -irregular paths in Φ of type III which originate from v . Since $\Phi(n-1)$ is U -folded then by Proposition 4.9 for each $r \in \mathcal{P}$ there exists a path p_r in Φ such that $o(p_r) = o(r)$, $t(p_r) = t(r)$, $\mu(p_r) = \overline{\mu(r)}$. So, let

$$\mathcal{P}' = \{p_r \mid r \in \mathcal{P}\}.$$

It is easy to see that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{P}' \neq \emptyset$. Observe that it is enough to construct doublings only for paths from \mathcal{P}' .

We say that $r \in \mathcal{P}'$ has level k if k is the minimal natural number such that $r \in \Phi(k)$. Hence \mathcal{P}' has a natural level structure, that is, all its elements are ranged according to their levels. Observe that levels of paths from \mathcal{P}' are bounded by $n - 1$.

Below we introduce the inductive procedure which transforms $\Phi''(n-1)$ into a U -folded graph Φ_0 . Hence all elements of \mathcal{P}' will be doubled in Φ_0 .

Let

$$\pi(u_n) = \pi(h_1)u_{n-1}^{\alpha_1}\pi(h_2) \cdots u_{n-1}^{\alpha_m}\pi(h_{m+1}).$$

Let $k = 0$.

All paths in \mathcal{P}' of level 1 are labeled by reduced words in X^\pm . Let p_1f be the initial subpath of d such that $\mu(p_1) = \pi(g)$, $g \in F(X)$, $\mu(f) = u_1^\alpha$, $\alpha \in \mathbb{Z}[t] - \mathbb{Z}$. We can assume that $u_1 \in F(X)$ and $\alpha > 0$.

(i) Suppose there exist only finitely many elements of \mathcal{P}' of level 1. Then let $r \in \mathcal{P}'$ be the path of level 1 with the maximal length. Hence we have $\overline{\mu(r)} = g \circ u_1^{k(r)} \circ c(r)$, where $c(r)$ is an initial segment of u_1 and all other paths from \mathcal{P}'

of level 1 are initial subpaths of r . We split f as $f_1 f_2 \cdots f_{k(r)+2}$ so that $\mu(f_i) = u_1$, $i \in [1, k(r)+1]$, $\mu(f_{k(r)+2}) = u_1^{\alpha-1}$ and add to d paths z_i , $i \in [1, k(r)+1]$ so that $o(z_i) = o(f_i)$, $t(z_i) = t(f_i)$, $\mu(z_i) = \pi(u_1)$. Finally, we fold r with $z_1 z_2 \cdots z_{k(r)+1}$.

(ii) Suppose there exist infinitely many elements of \mathcal{P}' of level 1. Since Φ is finite it follows that for any path r of level 1 we have $\overline{\mu(r)} = g \circ u_1^{i(r)+l(r)K} \circ c(r)$, where $i(r) \in [0, K-1]$, $K \in \mathbb{N}$, $l(r) \in \mathbb{N}$ and $c(r)$ is an initial segment of u_1 . Hence we add at $o(f)$ a loop $f_1 f_2 \cdots f_K$ such that $t(f_i) = o(f_{i+1})$, $i \in [1, K-1]$, $t(f_K) = o(f_1) = o(f)$, $\mu(f_i) = u_1$, $i \in [1, K]$ and a path $z_1 z_2 \cdots z_K$ such that $o(z_i) = o(f_i)$, $t(z_i) = t(f_i)$, $i \in [1, K]$, $\mu(z_i) = \pi(u_1)$, $i \in [1, K]$. Finally we fold $f_1 f_2 \cdots f_K$ with f and it is easy to see that now all paths from \mathcal{P}' of level 1 can be folded with $z_1 z_2 \cdots z_K$.

We can assume that all elements of \mathcal{P}' of level $n-2$ have been folded with d . Suppose $k = n-1$.

Let $p_1 f$ be the initial subpath of d such that $\mu(p_1) = \pi(h_1)$, $\mu(f) = u_{n-1}^{\alpha_1}$. We can assume that $\alpha_1 > 0$.

We can assume that there exists a path $r \in \mathcal{P}'$ of level $n-1$ such that $\overline{\mu(r)} = h_1 \circ u_{n-1}^{\theta(r)}$, $\alpha_1 \geq \theta(r) \gg 0$. Otherwise we are done by the induction hypothesis. Hence, $r = r_1 r_2$, where $\mu(r_1) = \pi(h_1)$, $\overline{\mu(r_2)} = u_{n-1}^\theta$ and $t(r_1)$ belongs to some u_{n-1} -component D of Φ . On the other hand, observe that for any path $r' \in \mathcal{P}'$ of level $n-1$ such that $\overline{\mu(r')} = h_1 \circ u_{n-1}^{\theta(r')}$, $\alpha_1 \geq \theta(r') \gg 0$ we have $r' = r'_1 r'_2$, where $\mu(r'_1) = \pi(h_1)$, $\overline{\mu(r'_2)} = u_{n-1}^{\theta(r')}$ and $t(r'_1)$ belongs to D . Hence we fold f with P_D and the resulting u_{n-1} -component D' is u_{n-1} -folded.

Next, we consider elements r of \mathcal{P}' of level $n-2$ such that $\overline{\mu(r)} = h_1 \circ u_{n-1}^{\alpha_1} \circ h'_2(r)$, where h'_2 is an initial segment of h_2 . In this case $r = r_1 r_2 r_3$, where $\mu(r_1) = \pi(h_1)$, $\overline{\mu(r_2)} = u_{n-1}^{\alpha_1}$, $\mu(r_3) = \pi(h'_2)$ and we can assume that $r_1 r_2$ is already folded with d . But to fold r_3 we can use the induction hypothesis and so on. In a finite number of steps we obtain the required graph Φ_0 . The claim is proved.

Claim 4.27. *Let Φ be a $(\mathbb{Z}[t], X)$ -graph such that $l(\Phi) = n$ and $U(\Phi) = \{u_1, \dots, u_n\}$. Adding finitely many edges to Φ one can transform it into a $(\mathbb{Z}[t], X)$ -graph Φ_0 such that*

- (a) Φ_0 satisfies (1)–(8),
- (b) if $v \in V(\Phi)$ and $v_0 \in V(\Phi_0)$ corresponds to v then $L(\Phi_0, v_0) = L(\Phi, v)$.

Proof of Claim 4.27. In view of Claim 4.20 we can assume that Φ satisfies (1)–(7) and $\Phi(n-1)$ is U -folded.

Let C be a u_n -component of Φ and $v \in V(C)$. In view of Claim 4.25 we can construct a doubling for any (u_n, δ) -irregular path of type I or II originating from v . Hence by Claims 4.22 and 4.23 all (u_n, δ) -irregular paths of type I or II originating from v are doubled too. At the same time, in view of Claim 4.26 we obtain a new graph Φ' for which all (u_n, δ) -irregular paths of any type originating from v are doubled. Finally, since the number of u_n -components is finite as well as the number

of vertices in them then after finitely many steps we obtain the required graph Φ_0 . The claim is proved.

By Claim 4.27, Γ can be transformed into a U -folded graph Δ . □

It can be seen easily from the Proposition 4.13 that properties of U -folded graphs are similar to the properties of folded X -graphs introduced in [4]. In [4] X -graphs are used for studying the structure of subgroups in free groups and for solving various algorithmic and combinatorial problems for free groups and their subgroups. In the next subsection we try to do the same for finitely generated subgroups of $F^{\mathbb{Z}[t]}$ using $(\mathbb{Z}[t], X)$ -graphs.

4.3. Membership problem for subgroups of $F^{\mathbb{Z}[t]}$

In Sec. 4.1 (see Lemma 4.2) we saw that any finite $(\mathbb{Z}[t], X)$ -graph defines a subgroup in $F^{\mathbb{Z}[t]}$. We will show now that the converse statement is also true.

Proposition 4.28. *Let H be a finitely generated subgroup of $F^{\mathbb{Z}[t]}$. Then there exists a U -folded $(\mathbb{Z}[t], X)$ -graph Γ and a vertex v of Γ such that $L(\Gamma, v) = H$.*

Proof. Since H is finitely generated then there are elements h_1, \dots, h_k in $F^{\mathbb{Z}[t]}$ which generate H . Since $F^{\mathbb{Z}[t]}$ is a union of the following infinite chain of groups:

$$F(X) = G_0 < G_1 < \dots < G_n < \dots,$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i , there exists a minimal natural number n such that $h_i \in G_{n+1}$ for all $i \in [1, k]$.

We define an $(\mathbb{Z}[t], X)$ -graph Γ_1 in the following way. Γ_1 is a wedge of k circles wedged at a vertex denoted v_1 . The i th circle is subdivided into n_i edges which are oriented and labeled by letters from

$$B = X^\pm \cup \{u^\alpha \mid u \in U, \alpha \in \mathbb{Z}[t] - \mathbb{Z}\}$$

so that the label of the i th circle (as read from v_1 to v_1) is precisely the word $\pi(h_i)$. Note that Γ_1 is connected.

For any cycle p at v_1 in Γ_1 we have $\overline{\mu(p)} \in L(\Gamma_1, v_1)$ by definition. So $H \subset L(\Gamma_1, v_1)$. The converse is obviously true — if $g \in L(\Gamma_1, v_1)$ then it can be realized as a reduced label of some cycle at v_1 in Γ_1 , so it can be obtained as a finite product of basic cycles labeled by $\pi(h_i)$. Thus we have $H = \langle h_1, \dots, h_k \rangle \subset L(\Gamma_1, v_1)$.

By Proposition 4.13, from Γ_1 by finitely many free and u -foldings, where $u \in \bigcup_{i=1}^k U(h_i)$ one can obtain a finite U -folded $(\mathbb{Z}[t], X)$ -graph Γ which is connected and there exists some vertex v in Γ which corresponds to v_1 in Γ_1 . By Lemma 4.3 we have $L(\Gamma_1, v_1) = L(\Gamma, v)$. □

Observe that Γ constructed in the proposition above is not unique in general. But all graphs associated with H define the same language which coincides with H .

Proposition 4.13 only states the existence of $(\mathbb{Z}[t], X)$ -graph Γ for H . But, in fact, the following result follows directly from the procedures described in the proofs of Lemmas 3.15, 3.17 and 3.18 and Proposition 4.13.

Proposition 4.29. *There is an algorithm which, given finitely many standard decompositions of elements h_1, \dots, h_k from $F^{\mathbb{Z}[t]}$, constructs a U -folded $(\mathbb{Z}[t], X)$ -graph Γ , such that $L(\Gamma, v) = \langle h_1, \dots, h_k \rangle$.*

The next result is a solution of membership problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Proposition 4.30. *Every finitely generated subgroup of $F^{\mathbb{Z}[t]}$ has a solvable membership problem. That is, there exists an algorithm which, given finitely many standard decompositions of elements g, h_1, \dots, h_k from $F^{\mathbb{Z}[t]}$, decides whether or not g belongs to the subgroup $H = \langle h_1, \dots, h_n \rangle$ of $F^{\mathbb{Z}[t]}$.*

Proof. We construct a U -folded $(\mathbb{Z}[t], X)$ -graph Γ , such that $L(\Gamma, v) = \langle h_1, \dots, h_k \rangle$ which is a finite algorithmic procedure by Proposition 4.29.

Let

$$F < H_1 < H_2 < \dots < H_n,$$

be the extension series for g , where $g \in H_n$ and H_{i+1} is obtained from H_i by a centralizer extension of a single element u_i and let $U(g) = \{u_1, \dots, u_n\}$. Then we check if $g \in L(\Gamma, v)$ using inductive argument based on $|U(g)|$.

$g \in H = L(\Gamma, v)$ if and only if there exists a reduced cycle p at v in Γ such that $g = \overline{\mu(p)}$. By Proposition 4.9 of Γ we can assume that $\mu(p) = \pi(g)$.

Let

$$\pi(g) = \pi(h_1)u_n^{\alpha_1}\pi(h_2) \cdots u_n^{\alpha_m}\pi(h_{m+1}).$$

If $|U(g)| = 0$, that is, $\pi(g)$ is a reduced word in $\{X \cup X^{-1}\}$ then we just try to “read” $\pi(g)$ in $\Gamma(0)$ starting at the vertex v — this can be done as shown in [4, Proposition 7.2].

Suppose there is an algorithm which “reads” a standard decomposition of an element $h \in H_{n-1}$, that is $|U(g)| < n$, starting from any point $v' \in V(\Gamma)$ and returns answer “yes” if there exists a path in Γ corresponding to $\pi(h)$ or “no” if such path does not exist. Then we apply this algorithm to v and $\pi(h_1)$. If we get “no” as a result then it means by Proposition 4.9 that there exists no path for h_1 starting at v in Γ and we stop — g does not belong to H . If we get “yes” as a result, that is, we have a path p_1 for $\pi(h_1)$ and if $t(p_1)$ belongs to some u_n -component C of Γ then we try to “read” $u_n^{\alpha_1}$ as follows. A pair $(P_C, H_{u_n}(t(p_1)))$ is associated with C , where P_C is a finite positively oriented path and $H_{u_n}(t(p_1))$ is a finitely generated free abelian group. By Lemma 3.18 it is enough to check if $u_n^{\alpha_1} \in \overline{\mu(q)} * H_{u_n}(t(p_1))$, where q is a reduced subpath of P_C . Hence, if we can find such q then it is unique and we proceed with h_2 . If there exists no such subpath of P_C , we stop — g does

not belong to H . In finitely many steps we either find out that $g \notin H$ on some intermediate step or construct a path p from v to some $v_1 \in V(\Gamma)$, which is labeled by $\pi(g)$. If we manage to find p then we check if $v = v_1$ which holds if and only if $g \in H$. \square

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