Algebra and Logic, Vol. 44, No. 3, 2005

BOUNDED ALGEBRAIC GEOMETRY OVER A FREE LIE ALGEBRA

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Keywords: *free Lie algebra, algebraic geometry, bounded algebraic set, coordinate algebra.*

Bounded algebraic sets over a free Lie algebra F over a field k are classified in three equivalent languages: (1) in terms of algebraic sets; (2) in terms of radicals of algebraic sets; (3) in terms of coordinate algebras of algebraic sets.

INTRODUCTION

The objective of the present paper is to form a groundwork for algebraic geometry over a free Lie algebra F over a field k. A basic task of algebraic geometry over an algebraic system is to describe algebraic sets over that system. As a rule, this problem is extremely difficult to tackle. In choosing a free Lie algebra F to be an algebraic system over a field k, therefore, the problem is confined to describing algebraic sets over F in terms of sets of the following two types:

(1) algebraic sets defined by systems of equations in one variable;

(2) bounded algebraic sets, that is, sets that lie inside an n-dimensional parallelepiped. (For a formal definition of n-parallelepipeds, see Sec. 2.)

In Sec. 6, we show that the classification of algebraic sets of the first type reduces to classifying sets of the second. In this paper, we are mainly concerned with examining the latter.

In classifying bounded algebraic sets over an algebra F, use will be made of three equivalent languages — namely, we describe them in terms of: (1) algebraic sets; (2) radicals of algebraic sets; (3) coordinate algebras of algebraic sets.

In Sec. 1, following [1, 2], we outline foundations of algebraic geometry over algebraic systems, and we cite some of the basic notions and results bearing on algebraic geometry over groups. Elements of algebraic geometry over Lie algebras were considered at length in [3]. Relevant results on algebraic geometry over metabelian Lie algebras can be found in [4-6].

In Sec. 2, we introduce the notion of an n-dimensional parallelepiped, and we show that this set from $F^n = \underbrace{F \times F \times \ldots \times F}_{n}$ is algebraic. In Sec. 3, we couch the definition of a bounded algebraic set, which is the main notion of our paper, and we look into the equivalence of different ways of defining this.

In Sec. 4, which takes center-stage in our account, we show how Diophantine geometry of a ground field k relates to algebraic geometry inside a fixed n-parallelepiped $\mathbb{V}$. We construct appropriate translators in the various languages. The basic, generalizing result is Theorem 4.7, which gives an exact description of

*Supported by RFBR grant No. 02-01-00192a.

Translated from *Algebra i Logika*, Vol. 44, No. 3, pp. 269-304, May-June, 2005. Original article submitted April 20, 2004.

the interplay between algebraic geometry inside $\mathbb{V}$ and Diophantine geometry of an affine space k^m, where m is a natural number, depending on $\mathbb{V}$.

In Sec. 5, we develop the idea of there being a correspondence between bounded algebraic sets over algebras F and algebraic sets over a field k, but without specifying an parallelepiped $\mathbb{V}$ explicitly. The main result of this section is Theorem 5.3, which deals in the correspondence between objects of the categories $\mathcal{B}AS(F)$ and $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$. It turns out that algebraic geometry over an algebra F is not simpler than that over a field k, and includes the latter as its part. Bounded algebraic sets over F are most simple to describe for the case of a finite field k — these are all possible combinations of points.

In the final Sec, 6, we describe algebraic sets over a free Lie algebra F, defined by systems of equations in one variable (briefly, in dimension one). In Theorem 6.2, it is shown that these are either bounded sets or the whole algebra F. In conclusion we give a list of the properties of a free Lie algebra used in our account. It is not hard to verify that those hold true also for a free anticommutative algebra, hence all the results of the present paper hold for it, too. (Pertinent information on Lie algebras, in particular, on a free Lie algebra F, can be found in [7, 8].)

1. ELEMENTS OF ALGEBRAIC GEOMETRY

In this section we give basic facts concerning algebraic geometry over Lie algebras. Let k be any field and let A be a fixed Lie algebra over k. In constructing the algebraic geometry over A, we assume that the elements of A are coefficients involved in representations of equations. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of unknowns. An algebra $A[X] = A * F(X)$ is called a *free A-algebra*, generated by the alphabet X. Here, $F(X)$ is a free Lie algebra generated by the set X, and $*$ stands for a free Lie product of Lie algebras. Elements like

$$f = f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, a_1, \ldots, a_r) \in A[X],$$

where $a_1, \ldots, a_r \in A$ are constants, are said to be *polynomials* in variables $x_1, \ldots, x_n$ with coefficients from A. Equating the polynomial f to zero yields an *equation* over A. An arbitrary subset S of the algebra $A[X]$ is called a *system of equations* over A. In the present paper, solutions for the equations $f \in A[X]$ are searched for in the algebra A proper — this is the so-called *Diophantine geometry* over A.

An *affine n-dimensional space* over A is the set

$$A^n = \{(b_1, \ldots, b_n) \mid b_i \in A\}.$$

Point $p = (b_1, \ldots, b_n) \in A^n$ is called the *root of a polynomial* $f \in A[X]$ if

$$f(p) = f(b_1, \ldots, b_n, a_1, \ldots, a_r) = 0.$$

Similarly, point $p \in A^n$ is called the *root* (*solution*) *of a system* $S \subseteq A[X]$ if p is a root of every polynomial in S.

An *algebraic set* over an algebra A solving S is the set

$$V(S) = \{p \in A^n \mid f(p) = 0 \ \ \forall\, f \in S\}.$$

Two systems of equations, S_1 and S_2, are said to be *equivalent* if $V(S_1) = V(S_2)$. System S is *inconsistent* over A if $V(S) = \varnothing$. Let $Y \subseteq A^n$. Then the *radical of a set* Y is the set

$$\mathrm{Rad}(Y) = \{f \in A[X] \mid f(p) = 0 \ \ \forall\, p \in Y\}.$$

Obviously, the radical of every set is an ideal of the algebra $A[X]$. We also speak of radicals of systems of equations. If $S \subseteq A[X]$ then $\mathrm{Rad}(S) = \mathrm{Rad}(V(S))$. Thus the radical of a system S consists of all polynomials that vanish at all solutions for S. A polynomial $f \in A[X]$ is called a *consequence* of S if $f \in \mathrm{Rad}(S)$. Alternatively, f is a consequence of S iff the system $S' = S \cup \{f\}$ is equivalent to S. In other words, $\mathrm{Rad}(S)$ is a maximal system of equations equivalent to S. If, however, S is inconsistent, then its radical is the whole algebra $A[X]$ by definition. The radical of an algebraic set defines that set uniquely. Alternatively, if $Y_1, Y_2 \subseteq A^n$ are algebraic sets, then

$$Y_1 = Y_2 \Leftrightarrow \mathrm{Rad}(Y_1) = \mathrm{Rad}(Y_2).$$

The next notion that is of importance in algebraic geometry is that of a coordinate algebra. The factor algebra

$$\Gamma(Y) = \Gamma(S) = A[X]/\mathrm{Rad}(Y)$$

is called the *coordinate algebra* of an algebraic set Y (or system S, $Y = V(S)$). As distinct from the radical, the coordinate algebra defines an algebraic set only up to isomorphism. (See below for definition of isomorphism between algebraic sets.)

The main goal of algebraic geometry over an algebra A is describing algebraic sets over A. As noted, this can be done in three equivalent languages:

(1) in terms of algebraic sets;

(2) in terms of radicals of algebraic sets;

(3) in terms of coordinate algebras of algebraic sets.

A proof that these approaches are equivalent can be found in [1] or [3]. In the present account the basic problem of algebraic geometry over a free Lie algebra is treated in all the three languages.

Categories. A basic result of algebraic geometry over any algebraic system is an equivalence theorem for categories of algebraic sets and their coordinate algebras. The category of coordinate algebras is a subcategory in the category of Lie A-algebras. Recall that a Lie algebra B over a field k is called an A-*algebra* if it contains a distinguished subalgebra isomorpic to A. A homomorphism $\varphi : B_1 \to B_2$ between A-algebras B_1 and B_2 is called an A-*homomorphism* if $\varphi(a) = a$ for all $a \in A$. The class of Lie A-algebras form a category whose morphisms are A-homomorphisms. Obviously, the coordinate algebra of any consistent system $S \subset A[X]$ of equations is a Lie A-algebra. (If S is inconsistent then $\Gamma(S) = 0$.) Thus the coordinate algebras of the non-empty algebraic sets over A form a complete subcategory in the category of all Lie A-algebras. We denote this subcategory by $\mathcal{C}A(A)$.

Objects of the category of algebraic sets are all possible algebraic sets over A. Morphisms are defined via polynomial maps. More precisely, for algebraic sets $Y \subseteq A^n$ and $Z \subseteq A^d$, the map $\phi : Y \to Z$ is a morphism in the category of algebraic sets if there exist polynomials $f_1, \ldots, f_d \in A[x_1, \ldots, x_n]$ such that for any point $(b_1, \ldots, b_n) \in Y$,

$$\phi(b_1, \ldots, b_n) = (f_1(b_1, \ldots, b_n), \ldots, f_d(b_1, \ldots, b_n)) \in Z.$$

Algebraic sets Y and Z are said to be *isomorphic* if there exist counter-morphisms $\phi : Y \to Z$ and $\theta : Z \to Y$ such that $\theta\phi = 1_Y$ and $\phi\theta = 1_Z$. The category of algebraic sets is denoted by $\mathcal{AS}(A)$.

THEOREM 1.1 (on the equivalence of categories of algebraic sets and coordinate algebras). The category $\mathcal{AS}(A)$ of algebraic sets over a Lie algebra A is equivalent to the category $\mathcal{C}A(A)$ of coordinate Lie A-algebras. In particular, the algebraic sets Y and Z over A are isomorphic if and only if their coordinate algebras are A-isomorphic, that is, $\Gamma(Y) \cong_A \Gamma(Z)$.

For proofs of the theorems given in the present section, we ask the reader to consult [1] or [3]. The correspondence stated in the equivalence theorem for categories is constructed using two contravariant functors. In so doing, with every algebraic set we associate its coordinate algebra, and given a coordinate algebra, its associated algebraic set is reconstructed, up to isomorphism, in the category of algebraic sets.

Coordinate algebras. Ambiguity in reconstructing an algebraic set from a coordinate algebra stems from the fact that the coordinate algebra is defined as some factor algebra. To do away with this, for instance, we can pass from the abstract definition of a coordinate algebra $\Gamma(Y)$ to its concrete representation, $\Gamma(Y) = \langle X, R\rangle_A$, in the category of Lie A-algebras, using X the set of generating elements and $R = \mathrm{Rad}(Y)$ the set of defining relations.

In this paper, we use a quite definite realization of coordinate algebras in some algebra $\bar{A}$ (for definition, see below). Under such a realization, the relationship between Y and $\Gamma(Y)$ is defined more explicitly.

A representation for coordinate algebras given below is obtained from the following representation of the radical $\mathrm{Rad}(S)$ of an algebraic set $V(S)$:

$$\mathrm{Rad}(S) = \bigcap_{p \in V(S)} \mathrm{Ker}\, \varphi_p,$$

where $\varphi_p : A[X] \to A$ is an A-homomorphism computing polynomials at a fixed point $p \in A^n$, which acts by the rule

$$f \in A[X], \quad \varphi_p(f) = f(p) \in A,$$

with $A[X]/\mathrm{Ker}\, \varphi_p \cong A$.

Denote by $\bar{A} = \prod_{i \in I} A^{(i)}$ a Cartesian product of copies of A, in which the index set I has cardinality $\max\{\aleph_0, |A|\}$. $\bar{A}$ is an A-algebra, and we assume that A is embedded in the Cartesian product $\bar{A}$ diagonally.

THEOREM 1.2. Let $Y \subseteq A^n$ be an algebraic set over an algebra A. Then the coordinate algebra $\Gamma(Y)$ is A-embedded in the algebra $\bar{A}$. Conversely, every finitely generated A-subalgebra of $\bar{A}$ is a coordinate algebra, for some algebraic set over A.

Theorem 1.2 yields a realization of coordinate algebras in the Cartesian product $\prod_{i \in I} A^{(i)}$, which will be made use of below. We re-word the results of Theorems 1.1 and 1.2 so as to tailor them to the form which will be more suitable for our further reasoning (Thm. 1.3).

Definition. Let $Y \subseteq A^n$ be an algebraic set over a Lie algebra A. We say that an n-generated A-subalgebra of $\bar{A} = \prod_{i \in I} A^{(i)}$ such as

$$C = \langle A, x_1, \ldots, x_n\rangle, \quad x_1, \ldots, x_n \in \bar{A},$$

is a *realization* of the coordinate algebra $\Gamma(Y)$ in the algebra $\bar{A}$ if the complete set $R \subset A[X]$ of relations on generators $x_1, \ldots, x_n \in \bar{A}$ coincides with $\mathrm{Rad}(Y)$.

In the definition above, it is essential that the number n of generators coincides with the dimension of an affine space A^n, in which Y is realized itself, and, moreover, the generators $x_1, \ldots, x_n \in \bar{A}$ are fixed so that $R = \mathrm{Rad}(Y)$. However, even these restrictions fail to guarantee that the coordinate algebra $\Gamma(Y)$ will be uniquely realized in the Cartesian product $\bar{A}$. Namely, we have

Remark. Let $C = \langle A, x_1, \ldots, x_n\rangle$ and $\tilde{C} = \langle A, \tilde{x}_1, \ldots, \tilde{x}_n\rangle$ be n-generated A-subalgebras of $\bar{A}$, and let $R, \tilde{R} \subset A[X]$ be the complete relation sets on generators $x_1, \ldots, x_n$ and $\tilde{x}_1, \ldots, \tilde{x}_n$, respectively. The algebras C and $\tilde{C}$ are realizations in $\bar{A}$ of the coordinate algebra of the same algebraic set iff $R = \tilde{R}$.

THEOREM 1.3. Let $Y \subseteq A^n$ be an algebraic set of a Lie algebra A. Then its coordinate algebra $\Gamma(Y)$ has a realization as a finitely generated A-subalgebra of the algebra $\prod\limits_{i \in I} A^{(i)}$, that is,

$$\Gamma(Y) = \langle A, x_1, \ldots, x_n \rangle, \quad x_1, \ldots, x_n \in \prod_{i \in I} A^{(i)},$$

such that $x_1, \ldots, x_n$ can be chosen so that $Y = \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$.

Conversely, if $C = \langle A, x_1, \ldots, x_n \rangle$, $x_1, \ldots, x_n \in \bar{A}$, is a finitely generated A-subalgebra of $\bar{A} = \prod\limits_{i \in I} A^{(i)}$, then there exists a unique algebraic set Y over A such that C is a realization of the coordinate algebra $\Gamma(Y)$ in $\bar{A}$. In this case $Y \supseteq \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$.

2. ALGEBRAIC SETS: PARALLELEPIPEDS

Let F be a free finitely generated Lie algebra over a field k. We write $a \circ b$ to denote the product of elements $a, b \in F$. By writing $a_1 \circ a_2 \circ \ldots \circ a_n$ we mean a product of the elements $a_1, a_2, a_3, \ldots, a_n$ with a left-normed placement of parentheses such as

$$(\ldots((a_1 \circ a_2) \circ a_3) \circ \ldots) \circ a_n.$$

Multiplication of $a \in F$ by coefficients in k is denoted αa, $\alpha \in k$, or $\alpha \cdot a$. First, we present some examples of algebraic sets over F.

Finite-dimensional affine spaces. We claim that every finite-dimensional linear subspace V of F is an algebraic set, that is, a solution for the equation $s(x) = 0$ in one variable x over F. We give a detailed description of the equations whose solutions are zero-, one-, and two-dimensional subspaces in F.

A 0-ary subspace is a solution for $s_0(x) = x = 0$. We take an arbitrary non-zero element $v_1 \in F$. Then the solution for $s_1(x) = x \circ v_1 = 0$ is, as is known, a one-dimensional subspace $V = \mathrm{lin}_k\{v_1\}$.

Now, let $v_1, v_2 \in F$ be a linearly independent pair of elements and let $V = \mathrm{lin}_k\{v_1, v_2\}$. We construct an equation $s_2(x) = (x \circ v_1) \circ (v_2 \circ v_1) = 0$. To do this, we verify that $V = V(s_2)$. Obviously, all elements of V satisfy $s_2(x) = 0$. The element $v_2 \circ v_1$ is not equal to zero since v_1 and v_2 are linearly independent. If v is a solution for $s_2(x)$ then $v \circ v_1 = \alpha_2(v_2 \circ v_1)$ for some $\alpha_2 \in k$. Consequently, $(v - \alpha_2 v_2) \circ v_1 = 0$, and hence $v - \alpha_2 v_2 = \alpha_1 v_1$ for some $\alpha_1 \in k$. Thus $v = \alpha_1 v_1 + \alpha_2 v_2$, that is, $v \in V$.

For the case of many-dimensional linear subspaces, we introduce the following notation:

$s_1(x) = s_1(x, v_1) = x \circ v_1$ is an equation over F, depending on a variable x and on a constant $v_1 \in F$;

$s_2(x) = s_2(x, v_1, v_2) = s_1(x, v_1) \circ s_1(v_2, v_1) = (x \circ v_1) \circ (v_2 \circ v_1)$ is an equation depending on two constants $v_1, v_2 \in F$;

$s_3(x) = s_3(x, v_1, v_2, v_3) = s_2(x) \circ s_2(v_3) = ((x \circ v_1) \circ (v_2 \circ v_1)) \circ ((v_3 \circ v_1) \circ (v_2 \circ v_1))$;

$\ldots$

$s_m(x) = s_m(x, v_1, \ldots, v_m) = s_{m-1}(x) \circ s_{m-1}(v_m) = s_{m-1}(x, v_1, \ldots, v_{m-1}) \circ s_{m-1}(v_m, v_1, \ldots, v_{m-1})$;

$\ldots.$

All equations $s_m(x) = 0$ have, in their representations, one occurrence of the variable x and are therefore linear in x. Furthermore, $s_m(0) = 0$. Hence solutions for $s_m(x) = 0$ are liner subspaces in F.

LEMMA 2.1. Let $v_1, \ldots, v_m \in F$ be linearly independent elements. Then

$$\mathrm{lin}_k\{v_1, \ldots, v_m\} = V(s_m), \quad s_m(x) = s_m(x, v_1, \ldots, v_m).$$

The **proof** is by induction on m. The base of induction was verified above. We show that $\mathrm{lin}_k\{v_1, \ldots, v_m\} \subseteq V(s_m)$. The equation $s_m(x) = 0$ is linear in x, and so we need only show that $v_1, \ldots, v_m \in V(s_m)$. By the inductive assumption, $s_{m-1}(v_i) = 0$ for $i = 1, \ldots, m-1$. Hence $s_m(v_i) = s_{m-1}(v_i) \circ s_{m-1}(v_m) = 0$, $i = 1, \ldots, m-1$. For $i = m$, we have $s_m(v_m) = s_{m-1}(v_m) \circ s_{m-1}(v_m) = 0$. Thus $\mathrm{lin}_k\{v_1, \ldots, v_m\} \subseteq V(s_m)$.

Now, we take an arbitrary element $v \in V(s_m)$ and show that $v \in \mathrm{lin}_k\{v_1, \ldots, v_m\}$. Since $v_m \notin \mathrm{lin}_k\{v_1, \ldots, v_{m-1}\}$, and by the inductive assumption, $s_{m-1}(v_m) \neq 0$, it follows that $s_m(v) = s_{m-1}(v) \circ s_{m-1}(v_m) = 0$ implies $s_{m-1}(v) = \alpha_m s_{m-1}(v_m)$ for some $\alpha_m \in k$. In view of the fact that $s_{m-1}(x)$ is linear, we conclude that $s_{m-1}(v - \alpha_m v_m) = 0$. By the inductive assumption, $v - \alpha_m v_m \in \mathrm{lin}_k\{v_1, \ldots, v_{m-1}\}$, and hence $v \in \mathrm{lin}_k\{v_1, \ldots, v_m\}$. $\square$

Thus all finite-dimensional spaces in F are algebraic sets. Moreover, affine translations of such spaces will be algebraic sets as well. We take linearly independent elements $v_1, \ldots, v_m \in F$ and any element $c \in F$. By Lemma 2.1, $V = \mathrm{lin}_k\{v_1, \ldots, v_m\}$ is an algebraic set, that is, it consists of solutions for $s_m(x) = 0$. Consequently, the affine subspace $V + c$ is an algebraic set solving $s_m(x - c) = 0$.

Parallelepipeds. Generalizing the results above to the case of systems of equations in several variables, we obtain

Proposition 2.2. Let V_i, $i = 1, \ldots, n$, be finite-dimensional linear subspaces of an algebra F, and let $c_1, \ldots, c_n \in F$ be arbitrary elements. Then the Cartesian product $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n) \subset F^n$ of affine spaces is an algebraic set over F. The set $\mathbb{V}$ solves a splitting system S of equations in n variables, that is,

$$
\begin{gathered}
S = \{s_{m_1}(\bar{x}), \ldots, s_{m_n}(\bar{x})\}, \\
s_{m_1}(\bar{x}) = s_{m_1}(x_1 - c_1, x_2, \ldots, x_n) = s_{m_1}(x_1 - c_1), \quad m_1 = \dim_k V_1, \\
\cdots \\
s_{m_n}(\bar{x}) = s_{m_n}(x_1, x_2, \ldots, x_n - c_n) = s_{m_n}(x_n - c_n), \quad m_n = \dim_k V_n.
\end{gathered}
$$

Sets such as $\mathbb{V}$ in Proposition 2.2 are called *n-parallelepipeds*. These will play an important role in our further reasoning. We will often speak of a particular n-parallelepiped, which requires that we have knowledge as to the ranks and bases for the affine spaces $V_1 + c_1, \ldots, V_n + c_n$. We give some relevant notation.

By $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n) \subset F^n$ we denote an n-parallelepiped, where $V_i + c_i$, $i = 1, \ldots, n$, are finite-dimensional affine subspaces of F, and namely,

$$
\begin{gathered}
V_1 + c_1: \ V_1 = \mathrm{lin}_k\{v_1^1, \ldots, v_{m_1}^1\}, \ \dim_k V_1 = m_1, \ c_1 \in F, \\
V_2 + c_2: \ V_2 = \mathrm{lin}_k\{v_1^2, \ldots, v_{m_2}^2\}, \ \dim_k V_2 = m_2, \ c_2 \in F, \\
\cdots \\
V_n + c_n: \ V_n = \mathrm{lin}_k\{v_1^n, \ldots, v_{m_n}^n\}, \ \dim_k V_n = m_n, \ c_n \in F.
\end{gathered}
$$

By Proposition 2.2, $\mathbb{V}$ is an algebraic set over a free Lie algebra F. The radical $\mathrm{Rad}(\mathbb{V})$ is generated by Lie polynomials $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n)$, which distinguish affine spaces $V_1 + c_1, \ldots, V_n + c_n$, respectively. For a deeper insight into the n-parallelepiped $\mathbb{V}$, we also look at its coordinate algebra. In this section we give some preliminary information about the coordinate algebra $\Gamma(\mathbb{V})$, which will be described at length in Sec. 4 (Prop. 4.9).

Put $\bar{F} = \prod_{i \in I} F^{(i)}$, which is a Cartesian product of the copies for F, where the index set I has cardinality

$|F|$, $\bar{k} = \prod_{i\in I} k^{(i)}$. By Theorem 1.3, $\Gamma(\mathbb{V})$ has a realization as an n-generated F-subalgebra of $\bar{F}$, and namely,

$$\Gamma(\mathbb{V}) = \langle F, x_1, \ldots, x_n\rangle, \quad x_1, \ldots, x_n \in \bar{F}.$$

Proposition 2.3. Let $\Gamma(\mathbb{V}) = \langle F, x_1, \ldots, x_n\rangle$ be any realization of the coordinate algebra $\Gamma(\mathbb{V})$ for an n-parallelepiped $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n)$ in the algebra $\bar{F} = \prod_{i\in I} F^{(i)}$. Then the generators $x_1, \ldots, x_n \in \bar{F}$ are representable as

$$\begin{gathered} x_1 = t_1^1 v_1^1 + \ldots + t_{m_1}^1 v_{m_1}^1 + c_1, \quad t_1^1, \ldots, t_{m_1}^1 \in \bar{k}, \\ \ldots \\ x_n = t_1^n v_1^n + \ldots + t_{m_n}^n v_{m_n}^n + c_n, \quad t_1^n, \ldots, t_{m_n}^n \in \bar{k}. \end{gathered}$$

Proof. By Theorem 1.3, for any realization $\Gamma(\mathbb{V}) = \langle F, x_1, \ldots, x_n\rangle$, we have $\mathbb{V} \supseteq \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$. Hence $x_1, \ldots, x_n \in \bar{F}$ have the required representations. $\square$

3. BOUNDED ALGEBRAIC SETS

Definition. An algebraic set $Y \subset F^n$ is said to be *bounded* if Y is contained in some n-parallelepiped. Clearly, all n-parallelepipeds are bounded sets, and if Y is a bounded set then there exist infinitely many n-parallelepipeds containing Y.

The lemma below translates the definition of bounded algebraic sets into the language of radical ideals.

LEMMA 3.1. An algebraic set $Y \subset F^n$ is bounded if and only if the radical $\mathrm{Rad}(Y)$ contains n Lie polynomials of the form $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n)$.

Proof. In fact, an algebraic set Y is contained in an n-parallelepiped $\mathbb{V}$ iff $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n) \in \mathrm{Rad}(Y)$. $\square$

In the category $\mathcal{AS}(F)$ of all algebraic sets over F, we define a complete subcategory $\mathcal{B}AS(F)$, whose objects are bounded algebraic sets only.

LEMMA 3.2. Let $Y \subset F^n$ and $Z \subset F^d$ be two algebraic sets over an algebra F for which there exists an epimorphism $\phi : Y \to Z$, and Y is bounded. Then Z is also bounded.

Proof. Assume that $\mathbb{V}$ is an n-parallelepiped containing Y. Denote by $f_1, \ldots, f_d \in F[x_1, \ldots, x_n]$ the Lie polynomials defining ϕ, that is, for every point $(b_1, \ldots, b_n) \in Y$,

$$\phi(b_1, \ldots, b_n) = (f_1(b_1, \ldots, b_n), \ldots, f_d(b_1, \ldots, b_n)) \in Z.$$

Since $Y \subseteq \mathbb{V}$, we can write

$$\begin{gathered} b_1 = \alpha_1^1 v_1^1 + \ldots + \alpha_{m_1}^1 v_{m_1}^1 + c_1, \quad \alpha_1^1, \ldots, \alpha_{m_1}^1 \in k, \\ \ldots \\ b_n = \alpha_1^n v_1^n + \ldots + \alpha_{m_n}^n v_{m_n}^n + c_n, \quad \alpha_1^n, \ldots, \alpha_{m_n}^n \in k. \end{gathered}$$

Substituting $p = (b_1, \ldots, b_n) \in Y$ in the Lie polynomials $f_1, \ldots, f_d$ yields

$$\begin{gathered} f_1(p) = \beta_1^1 w_1^1 + \ldots + \beta_{l_1}^1 w_{l_1}^1 + r_1, \quad \beta_1^1, \ldots, \beta_{l_1}^1 \in k, \\ \ldots \\ f_d(p) = \beta_1^d w_1^d + \ldots + \beta_{l_d}^d w_{l_d}^d + r_d, \quad \beta_1^d, \ldots, \beta_{l_d}^d \in k, \end{gathered}$$

where $w_1^1, \ldots, w_{l_1}^1, r_1, \ldots, w_1^d, \ldots, w_{l_d}^d, r_d \in F$ are elements not depending on a point $p \in Y$. Only the coefficients $\beta_1^1, \ldots, \beta_{l_1}^1, \ldots, \beta_1^d, \ldots, \beta_{l_d}^d \in k$ depend on it. Let $\mathbb{W} = (W_1 + r_1) \times \ldots \times (W_d + r_d)$ be a d-parallelepiped, where $W_1 = \mathrm{lin}_k\{w_1^1, \ldots, w_{l_1}^1\}, \ldots, W_d = \mathrm{lin}_k\{w_1^d, \ldots, w_{l_d}^d\}$. It is clear that $\phi(Y) \subseteq \mathbb{W}$. Since ϕ is an epimorphism, $\phi(Y) = Z$; hence, Z is contained in $\mathbb{W}$ and Z is bounded. □

COROLLARY. Let Y and Z be isomorphic algebraic sets over F. If Y is a bounded set then Z is also a bounded algebraic set. Therefore the category $\mathcal{B}AS(F)$ of bounded algebraic sets is closed under isomorphisms.

The description of algebraic sets inside $\mathbb{V}$ depends on a ground field k and is presented in Sec. 4.

Coordinate algebras of bounded algebraic sets. Denote by $B(\bar{F})$ a subalgebra in $\bar{F} = \prod\limits_{i \in I} F^{(i)}$, $|I| = |F|$, consisting of elements the degrees of coordinates of which are bounded in totality. We say that $B(\bar{F})$ is a *bounded* subalgebra. The next lemma delivers an extra characteristic for it.

LEMMA 3.3. The algebra $B(\bar{F})$ is isomorphic to a tensor product $B(\bar{F}) \cong_F F \otimes_k \bar{k}$, where $\bar{k} = \prod\limits_{i \in I} k^{(i)}$ is the Cartesian product of copies of a field k of cardinality $|F|$.

Proof. There is a natural F-embedding $\varphi : F \otimes_k \bar{k} \to \bar{F}$, under which any element

$$t_1 v_1 + \ldots + t_m v_m \in F \otimes_k \bar{k}, \quad v_1, \ldots, v_m \in F, \quad t_1, \ldots, t_m \in \bar{k},$$

is mapped to

$$w = t_1 v_1 + \ldots + t_m v_m \in \bar{F}, \quad w^{(i)} = t_1^{(i)} v_1 + \ldots + t_m^{(i)} v_m, \quad i \in I.$$

Clearly, $\varphi(F \otimes_k \bar{k}) \subseteq B(\bar{F})$ in this instance.

Conversely, let $w \in B(\bar{F})$ and let the degrees of all elements $w^{(i)}$, $i \in I$ (coordinates of w), not exceed n. Then, in particular, every coordinate $w^{(i)}$ is a linear combination over k of regular Hall monomials of degree at most n, whose number is finite. In other words, there are elements $v_1, \ldots, v_m \in F$ such that $w^{(i)} = \alpha_1^{(i)} v_1 + \ldots + \alpha_m^{(i)} v_m$ for some $\alpha_j^{(i)} \in k$, $i \in I$. It follows that $w = t_1 v_1 + \ldots + t_m v_m$, where $t_1, \ldots, t_m \in \bar{k}$, that is, w has a preimage in the tensor product $F \otimes_k \bar{k}$. □

Theorem 1.2 implies that all coordinate algebras of algebraic sets over F are F-subalgebras in the Cartesian product $\bar{F} = \prod\limits_{i \in I} F^{(i)}$. In this case a same coordinate algebra has different realizations in $\bar{F}$ (see Thm. 1.3).

Definition. Let Y be an algebraic set over F. The coordinate algebra $\Gamma(Y)$ is said to be *bounded* if there exists its realization in $\bar{F}$, lying in a bounded subalgebra $B(\bar{F})$.

The fact that the property of being bounded for $\Gamma(Y)$ does not depend on its particular realization is exemplified by the following:

LEMMA 3.4. Let C_1 and C_2 be F-isomorphic finitely generated F-subalgebras of $\bar{F}$, written $C_1 \cong_F C_2$, such that C_1 is a subalgebra of a bounded algebra $B(\bar{F})$. Then C_2 is also a subalgebra of $B(\bar{F})$.

Proof. Let $\varphi : C_2 \to C_1$ be an F-isomorphism. We take an arbitrary element $w \in C_2$ and show that $w \in F \otimes_k \bar{k}$. Since $\varphi(w) \in C_1$ and $C_1 \subset F \otimes_k \bar{k}$, there exist elements $v_1, \ldots, v_m \in F$ for which $\varphi(w) = t_1 v_1 + \ldots + t_m v_m$, $t_j \in \bar{k}$, $j = 1, \ldots, m$. It follows that $w = \tilde{t}_1 v_1 + \ldots + \tilde{t}_m v_m$ for some $\tilde{t}_j \in \bar{k}$, $j = 1, \ldots, m$. Indeed, consider a Lie polynomial $s_m(x, v_1, \ldots, v_m)$, defined as in Sec. 2. Clearly, $s_m(\varphi(w)) = 0$, and hence $\varphi(s_m(w)) = 0$ and $s_m(w) = 0$. Consequently, for every $i \in I$, there are $\alpha_j^{(i)} \in k$ such that $w^{(i)} = \alpha_1^{(i)} v_1 + \ldots + \alpha_m^{(i)} v_m$, whence the result. □

COROLLARY 1. If C_1 and C_2 are two realizations of the coordinate algebra $\Gamma(Y)$ in an algebra $\bar{F}$, and, moreover, if C_1 is contained in a bounded subalgebra $B(\bar{F})$, then C_2 is also contained in $B(\bar{F})$.

Thus the coordinate algebra $\Gamma(Y)$ of an algebraic set Y over F is bounded iff any one of its realizations in $\bar{F}$ is contained in $B(\bar{F})$.

Using Theorem 1.1 we arrive at

COROLLARY 2. Let Y and Z be isomorphic algebraic sets over F. If the coordinate algebra $\Gamma(Y)$ is bounded then the coordinate algebra $\Gamma(Z)$ is also bounded.

We distinguish a complete subcategory, $\mathcal{B}CA(F)$, of bounded coordinate algebras in the category, $\mathcal{C}A(F)$, of all coordinate algebras of algebraic sets over F.

Proposition 3.5. An algebraic set $Y \subset F^n$ over a free Lie algebra F is bounded if and only if its coordinate algebra $\Gamma(Y)$ is bounded.

Proof. Let $\Gamma(Y) = \langle F, x_1, \ldots, x_n \rangle$. In view of Corollary 1 to Lemma 3.4, the property of being bounded for $\Gamma(Y)$ does not depend on the choice of its realization in $\bar{F}$. By Theorem 1.3, the generators $x_1, \ldots, x_n$ can be chosen so that $Y = \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$. It is clear that $\Gamma(Y)$ is bounded iff its generators $x_1, \ldots, x_n$ belong to $B(\bar{F})$, $B(\bar{F}) = F \otimes_k \bar{k}$.

Suppose Y is a bounded set, $Y \subseteq \mathbb{V}$, and $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n)$ is an n-parallelepiped. Then every point $(b_1, \ldots, b_n) \in Y$ has the form

$$b_1 = \alpha_1^1 v_1^1 + \ldots + \alpha_{m_1}^1 v_{m_1}^1 + c_1, \quad \alpha_1^1, \ldots, \alpha_{m_1}^1 \in k,$$
$$\ldots$$
$$b_n = \alpha_1^n v_1^n + \ldots + \alpha_{m_n}^n v_{m_n}^n + c_n, \quad \alpha_1^n, \ldots, \alpha_{m_n}^n \in k.$$

Consequently, $x_1 = \{b_1^{(i)} \mid i \in I\}, \ldots, x_n = \{b_n^{(i)} \mid i \in I\} \in F \otimes_k \bar{k}$.

Conversely, let $x_1, \ldots, x_n \in F \otimes_k \bar{k}$, so that

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \ldots, t_{m_i}^i \in \bar{k}, \quad i = 1, \ldots, n.$$

Then $(b_1, \ldots, b_n) \in \mathbb{V}$, that is, $Y \subseteq \mathbb{V}$. □

We conclude this section by describing the coordinate algebra $\Gamma(Y)$ of a given bounded algebraic set Y. In the proof of Proposition 3.5, we brought out some representation of $\Gamma(Y)$ in $\bar{F}$, but that was a particular realization for which $Y = \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$. Nevertheless, we can see that the description of $\Gamma(Y)$ is similar to that of the coordinate algebra for an n-parallelepiped (see Prop. 2.3).

Proposition 3.6. Let $Y \subset F^n$ be a bounded algebraic set over a free Lie algebra F, and let $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n)$ be some n-parallelepiped.

(1) If $Y \subseteq \mathbb{V}$ then, for any realization $\Gamma(Y) = \langle F, x_1, \ldots, x_n \rangle$ of the coordinate algebra $\Gamma(Y)$ in $\bar{F}$, the generators $x_1, \ldots, x_n$ have the following representation:

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \ldots, t_{m_i}^i \in \bar{k}, \quad i = 1, \ldots, n. \tag{i}$$

(2) If, for some realization $\Gamma(Y) = \langle F, x_1, \ldots, x_n \rangle$, $x_1, \ldots, x_n$ have a presentation by (i), then $Y \subseteq \mathbb{V}$.

Proof. Let $Y \subseteq \mathbb{V}$. By Theorem 1.3, $Y \supseteq \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$. It follows that $\mathbb{V} \supseteq \{(x_1^{(i)}, \ldots, x_n^{(i)}) \mid i \in I\}$, which yields the required representation for $x_1, \ldots, x_n$.

Now, suppose that the coordinate algebra $\Gamma(Y)$ has a realization $\Gamma(Y) = \langle F, x_1, \ldots, x_n \rangle$ such that the generators have a presentation by (i). Then the Lie polynomials $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n)$, distinguishing the respective affine spaces $V_1 + c_1, \ldots, V_n + c_n$, are relations on $x_1, \ldots, x_n \in \bar{F}$. Consequently, $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n) \in \operatorname{Rad}(Y)$ and $Y \subseteq \mathbb{V}$. □

Proposition 3.5 does not still yield a complete description of the coordinate algebra $\Gamma(Y)$ for a bounded set Y since nothing has been said as yet about the coefficients $t_1^1, \ldots, t_{m_1}^1, \ldots, t_1^n, \ldots, t_{m_n}^n \in \bar{k}$. The structure of $\Gamma(Y)$ will be described in more detail in Sec. 4 (see Prop. 4.8).

4. ALGEBRAIC GEOMETRY INSIDE PARALLELEPIPED

In this section, we thoroughly examine bounded algebraic sets over an algebra F in a fixed n-parallelepiped $\mathbb{V}$. The relationship is instituted between algebraic geometry over F inside $\mathbb{V}$ and Diophantine algebraic geometry over a ground field k.

We start by treating the case where $n = 1$. Fix a 1-parallelepiped $\mathbb{V}$, that is, a finite-dimensional affine space $\mathbb{V} = V + c \subset F$, where $V = \mathrm{lin}_k\{v_1, \ldots, v_m\}$ is a linear space over a field k of dimension m, $c \in F$. Below we show that there exists a one-to-one correspondence between algebraic sets over F, lying in $V + c$, and algebraic sets over k, lying in an affine space k^m, that is,

$$Y_F \subseteq V + c \leftrightarrow Y_k \subseteq k^m.$$

Correspondence of algebraic sets. First, we define the following two correspondences between algebraic sets:

(1) $Y_F \subseteq V + c \to Y_k \subseteq k^m$;

(2) $Y_k \subseteq k^m \to Y_F \subseteq V + c$.

(1) Let $Y_F \subseteq V + c$ be a bounded algebraic set. Put

$$Y_k = \{(\alpha_1, \ldots, \alpha_m) \in k^m \mid \alpha_1 v_1 + \ldots + \alpha_m v_m + c \in Y_F\}, \quad Y_k \subseteq k^m.$$

(2) Let Y_k be an algebraic set over a field k in dimension m. Put

$$Y_F = \{\alpha_1 v_1 + \ldots + \alpha_m v_m + c \mid (\alpha_1, \ldots, \alpha_m) \in Y_k\}, \quad Y_F \subseteq V + c.$$

The form of $Y_F \to Y_k$ and $Y_k \to Y_F$ is defined so that their composition, in any order, is identical. The fact that these correspondences associate algebraic sets with algebraic sets will be proved below, in Lemmas 4.1 and 4.2. Preparatory to this, we bring out some relevant facts.

Correspondence of radical ideals. As is known, every algebraic set is uniquely defined by its radical. In this case it seems convenient that the correspondences $Y_F \to Y_k$ and $Y_k \to Y_F$ between sets and similar correspondences between their radicals will be constructed in parallel.

Below, the radical $\mathrm{Rad}(Y_F)$ of a bounded algebraic set $Y_F \subseteq V + c$ is associated with a radical ideal $\mathrm{Rad}(S_k)$ of the ring $k[y_1, \ldots, y_m]$, and we prove that $\mathrm{Rad}(S_k)$ is exactly the radical of a set Y_k. Inversely, the radical $\mathrm{Rad}(Y_k)$ of a bounded algebraic set over a field k is associated with a radical ideal $\mathrm{Rad}(S_F)$ of the algebra $F[x]$, and we prove that $\mathrm{Rad}(S_F)$ is exactly the radical of a set Y_F.

We start to define correspondences between the radicals with a correspondence between individual polynomials in the algebra $F[x]$ and in the ring $k[y_1, \ldots, y_m]$. With an arbitrary Lie polynomial $f(x) \in F[x]$ we associate a finite system $S_f \subset k[y_1, \ldots, y_m]$ of equations such that

$$f(x) \in \mathrm{Rad}(Y_F) \Leftrightarrow S_f \subset \mathrm{Rad}(Y_k).$$

For clarity, we give a particular example. Let $V = \mathrm{lin}_k\{a_1, a_2\}$, where a_1 and a_2 are two distinct free generators of F, $c = 0$, and $Y_F \subseteq V$. Take a Lie polynomial $f(x) = (x \circ a_1) \circ x - (a_2 \circ a_1) \circ a_2 - (a_2 \circ a_1) \circ a_1$. Then

$$f(\alpha_1 a_1 + \alpha_2 a_2) = (\alpha_2^2 - 1) \cdot (a_2 \circ a_1) \circ a_2 + (\alpha_1 \alpha_2 - 1) \cdot (a_2 \circ a_1) \circ a_1.$$

System S_f is defined by setting

$$\{g_1(y_1, y_2) = y_2^2 - 1, \quad g_2(y_1, y_2) = y_1 y_2 - 1\}.$$

The elements $a_2a_1a_2,\ a_2a_1a_1 \in F$ are linearly independent (see [7, 8]). Therefore

$$f(\alpha_1 a_1 + \alpha_2 a_2) = 0 \Leftrightarrow g_1(\alpha_1, \alpha_2) = g_1(\alpha_1, \alpha_2) = 0.$$

The same argument fits in the general case. Namely, we take a polynomial $f(x) \in F[x]$ and exercise a substitution $x = p$, where

$$p = \alpha_1 v_1 + \ldots + \alpha_m v_m + c, \quad \alpha_1, \ldots, \alpha_m \in k,$$

assuming in so doing that the coefficients $\alpha_1, \ldots, \alpha_m$ are variables. Removing parentheses and collecting similar terms, we obtain

$$f(p) = g_1(\alpha_1, \ldots, \alpha_m)u_1 + \ldots + g_s(\alpha_1, \ldots, \alpha_m)u_s,$$

where $u_1, \ldots, u_s \in F$ are some linearly independent elements not depending on a point $p \in Y$, and $g_1, \ldots, g_p \in k[y_1, \ldots, y_m]$ are polynomials. Clearly, in this case

$$f(\alpha_1 v_1 + \ldots + \alpha_m v_m + c) = 0 \Leftrightarrow g_1(\alpha_1, \ldots, \alpha_m) = \ldots = g_s(\alpha_1, \ldots, \alpha_m) = 0.$$

Therefore we put

$$S_f = \{g_1, \ldots, g_s\} \subset k[y_1, \ldots, y_m].$$

We define the inverse correspondence. Consider any polynomial $g \in k[y_1, \ldots, y_m]$ such as

$$g(y_1, \ldots, y_m) = \sum_{\bar{i}} \alpha_{\bar{i}} y_1^{i_1} \cdot \ldots \cdot y_m^{i_m}, \quad \alpha_{\bar{i}} \in k, \quad \bar{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m.$$

Let $M_1 = \max\{i_1\}, \ldots, M_n = \max\{i_m\}$. With $g(\bar{y})$ we associate a Lie polynomial $f_g(x)$ so that

$$g(\bar{y}) \in \mathrm{Rad}(Y_k) \Leftrightarrow f_g(x) \in \mathrm{Rad}(Y_F).$$

In order to construct $f_g(x)$, we introduce some extra notation:

$f_m(x) = s_{m-1}(x - c, v_1, \ldots, v_{m-1})$, where s_{m-1} is a Lie polynomial from Sec. 2;

$f_{m-1}(x) = s_{m-1}(x - c, v_1, \ldots, v_{m-2}, v_m)$;

$\ldots$

$f_1(x) = s_{m-1}(x - c, v_2, \ldots, v_{m-1}, v_m)$.

Substituting in $f_1(x), \ldots, f_m(x)$ point $x = p$, where

$$p = \alpha_1 v_1 + \ldots + \alpha_m v_m + c, \quad \alpha_1, \ldots, \alpha_m \in k,$$

we arrive at

$$f_m(p) = \alpha_m b_m,\ f_{m-1}(p) = \alpha_{m-1} b_{m-1},\ \ldots,\ f_1(p) = \alpha_1 b_1,$$

where $b_1, \ldots, b_m \in F$ are non-zero elements.

We choose an element $a \in F$ so that the degree of a is greater than the degrees of all elements $b_1, \ldots, b_m$. Then all possible products of the form $ab_1 \ldots b_1 b_2 \ldots b_2 \ldots b_m \ldots b_m$ are distinct from zero.

The Lie polynomial $f_g(x)$ is defined thus:

$$f_g(x) = \sum_{\bar{i}} \alpha_{\bar{i}} a \circ \underbrace{f_1(x) \circ \ldots \circ f_1(x)}_{i_1} \circ \underbrace{b_1 \circ \ldots \circ b_1}_{M_1 - i_1} \circ \ldots \circ \underbrace{f_m(x) \circ \ldots \circ f_m(x)}_{i_m} \circ \underbrace{b_m \circ \ldots \circ b_m}_{M_m - i_m}.$$

Substituting point $x = p$ in $f_g(x)$ yields

$$f_g(p) = g(\alpha_1, \ldots, \alpha_m) \cdot a \circ \underbrace{b_1 \circ \ldots \circ b_1}_{M_1} \circ \ldots \circ \underbrace{b_m \circ \ldots \circ b_m}_{M_m} = g(\alpha_1, \ldots, \alpha_m) \cdot e,$$

where $e \in F$ is a non-zero element. This implies

$$f_g(\alpha_1 v_1 + \ldots + \alpha_m v_m + c) = 0 \Leftrightarrow g(\alpha_1, \ldots, \alpha_m) = 0.$$

We have thus defined the correspondences between individual polynomials of the algebra $F[x]$ and of the ring $k[y_1, \ldots, y_m]$.

Now, we define correspondences $\mathrm{Rad}(Y_F) \to \mathrm{Rad}(S_k)$ and $\mathrm{Rad}(Y_k) \to \mathrm{Rad}(S_F)$ between radical ideals, setting

$$\mathrm{Rad}(Y_F) \to S_k = \{S_f \mid f \in \mathrm{Rad}(Y_F)\},$$

$$\mathrm{Rad}(Y_k) \to S_F = \{f_g(x) \in F[x] \mid g \in \mathrm{Rad}(Y_k)\} \cup s_m(x - c, v_1, \ldots, v_m),$$

where $s_m(x - c, v_1, \ldots, v_m) = 0$ is an equation distinguishing an affine space $V + c$.

We come back to $Y_F \to Y_k$ and $Y_k \to Y_F$.

LEMMA 4.1. Let $Y_F \subseteq V + c$ be a bounded algebraic set over F, and let Y_k be the subset of an affine space k^m specified above. Then Y_k is an algebraic set over a field k and $Y_k = V(S_k)$, where the system $S_k \subseteq k[y_1, \ldots, y_m]$ is defined as above.

Proof. We claim that $Y_k = V(S_k)$. Indeed, point $(\alpha_1, \ldots, \alpha_m) \in k^m$ belongs to Y_k iff $p = \alpha_1 v_1 + \ldots + \alpha_m v_m + c$ belongs to Y_F. The inclusion $p \in Y_F$ is equivalent to the fact that $f(p) = 0$ for any $f(x) \in \mathrm{Rad}(Y_F)$. In turn, $f(p) = 0$ iff $g(\alpha_1, \ldots, \alpha_m) = 0$ for every $g \in S_f$. Consequently, $p \in Y_F$ iff $(\alpha_1, \ldots, \alpha_m) \in V(S_k)$. $\square$

LEMMA 4.2. Let $Y_k \subseteq k^m$ be an algebraic set over a field k, and let Y_F be the subset of an affine space $V + c$ specified above. Then Y_F is an algebraic set over F, with $Y_F = V(S_F)$, where the system $S_F \subseteq F[x]$ is defined as above.

Proof. We claim that $Y_F = V(S_F)$. Indeed, point $p = \alpha_1 v_1 + \ldots + \alpha_m v_m + c$ belongs to Y_F iff $(\alpha_1, \ldots, \alpha_m) \in Y_k$. The inclusion $(\alpha_1, \ldots, \alpha_m) \in Y_k$ is equivalent to the fact that $g(\alpha_1, \ldots, \alpha_m) = 0$ for every polynomial $g \in \mathrm{Rad}(Y_k)$. In turn, $g(\alpha_1, \ldots, \alpha_m) = 0$ iff $f_g(p) = 0$. Consequently, $(\alpha_1, \ldots, \alpha_m) \in Y_k$ iff $p \in V(S_F)$. $\square$

Thus the correspondences $Y_F \to Y_k$ and $Y_k \to Y_F$ associate algebraic sets with algebraic. Their composition in any order is identical, that is,

$$Y_F \to Y_k \to Y_F = \mathrm{id}_{AS(F)}, \quad Y_k \to Y_F \to Y_k = \mathrm{id}_{AS(k)}.$$

Thus we have in fact proved the following:

THEOREM 4.3. The mappings $Y_F \to Y_k$ and $Y_k \to Y_F$ determine a one-to-one correspondence $Y_F \leftrightarrow Y_k$ between algebraic subsets over F, lying in a finite-dimensional affine space $V + c \subset F$, $V = \mathrm{lin}_k\{v_1, \ldots, v_m\}$, $c \in F$, and algebraic sets over a field k, lying in an affine space k^m.

COROLLARY 1. $Y_F \leftrightarrow Y_k$ associates an affine space $V + c$ with an affine space k^m.

COROLLARY 2. $\mathrm{Rad}(Y_F) \leftrightarrow \mathrm{Rad}(Y_k)$ determines a one-to-one correspondence between radical ideals of the ring $k[y_1, \ldots, y_m]$ and radical ideals of the algebra $F[x]$ containing a Lie polynomial $s_m(x - c, v_1, \ldots, v_m)$.

An algorithm of determining this correspondence may be simplified in this way.

(1) Let $Y_F \subseteq V + c$ be a bounded algebraic set over F, and let $Y_F = V(S'_F)$, where S'_F is some system of equations. Put $S'_k = \{S_f \mid f \in S'_F\}$; then $S'_k \subseteq S_k$, but $\mathrm{Rad}(S'_k) = \mathrm{Rad}(S_k) = \mathrm{Rad}(Y_k)$.

The meaning of this simplification is that in order to define $\mathrm{Rad}(Y_k)$, it suffices to use only those Lie polynomials that generate $\mathrm{Rad}(Y_F)$.

(2) Let $Y_k \subseteq k^m$ be an algebraic set over k, and let $Y_k = V(S'_k)$, where S'_k is some system of equations. Put $S'_F = \{f_g(x) \mid g \in S'_k\} \cup s_m(x - c)$; then $S'_F \subseteq S_F$, but $\mathrm{Rad}(S'_F) = \mathrm{Rad}(S_F) = \mathrm{Rad}(Y_F)$.

COROLLARY 3. Let $Y_F \subseteq V + c$ be a bounded algebraic set over F, and let $S \subseteq F[x]$ be a system of equations for Y_F such that $Y_F = V(S)$. Then there exists a finite subsystem $S_0 \subseteq S$ for which $Y_F = V(S_0 \cup s_m(x - c, v_1, \ldots, v_m))$.

Proof. We construct the following system of equations over k:

$$S'_k = \{g \in k[y_1, \ldots, y_m] \mid g \in S_f,\ f \in S\}.$$

By Corollary 2, S'_k and S_k are equivalent. In view of k being Noetherian over equations, there exists a finite subsystem $S_{k,0} \subseteq S'_k$, equivalent to S'_k. Thus $Y_k = V(S_{k,0})$. For every polynomial $g \in S_{k,0}$, we fix a Lie polynomial $f \in S$ such that $g \in S_f$, and take the set of fixed polynomials to be S_0. It is easy to verify that point $p = \alpha_1 v_1 + \ldots + \alpha_m v_m + c$ belongs to $Y = V(S)$ iff $p \in V(S_0)$. Consequently, $Y_F = V(S_0 \cup s_m(x - c))$. $\square$

COROLLARY 4. If the ground field k is finite then any subset $M_F \subseteq V + c$ of an affine space $V + c$ is algebraic over F.

Proof. Indeed, for the case where k is a finite field, the subset $M_k \subseteq k^m$ is algebraic. $\square$

Correspondence of coordinate algebras. Above, we constructed one-to-one correspondences $Y_F \leftrightarrow Y_k$ and $\mathrm{Rad}(Y_F) \leftrightarrow \mathrm{Rad}(Y_k)$ between algebraic sets and between radical ideals. Now, we describe a similar correspondence in terms of coordinate algebras.

We assume that coordinate algebras of algebraic sets over F are realized in the Cartesian product $\bar{F} = \prod_{i \in I} F^{(i)}$, and coordinate rings of algebraic sets over k — in the Cartesian product $\bar{k} = \prod_{i \in I} k^{(i)}$, in which a diagonal subfield isomorphic to k is distinguished.

Definition. Let $Y_k \subseteq k^m$ be an algebraic set over a field k. We say that an m-generated k-subring of $\bar{k}$ such as

$$C_k = \langle k, t_1, \ldots, t_m \rangle, \quad t_1, \ldots, t_m \in \bar{k},$$

is a *realization* of the coordinate ring $\Gamma(Y_k)$ in the ring $\bar{k}$ if the complete relation set $R_k \subset k[y_1, \ldots, y_m]$ on generators $t_1, \ldots, t_m \in \bar{k}$ coincides with $\mathrm{Rad}(Y_k)$.

Let $Y_F \subseteq V + c$ be a bounded algebraic set over an algebra F. Proposition 3.6 says that for any realization $\Gamma(Y_F) = \langle F, x \rangle$, $x \in \bar{F}$, of the coordinate algebra $\Gamma(Y_F)$ in $\bar{F}$, the generator $x \in \bar{F}$ is representable as

$$x = t_1 v_1 + \ldots + t_m v_m + c \tag{$*$}$$

for some coefficients $t_1, \ldots, t_m \in \bar{k}$. Conversely, if $C_F = \langle F, x \rangle$ is some F-subalgebra of $\bar{F}$ with generator $x \in \bar{F}$ in the form $(*)$, then C_F is a realization of the coordinate algebra of a bounded algebraic set Y_F over

F, with $Y_F \subseteq V + c$. Let

$$R_F = \{f(x) \in F[x] \mid f(t_1 v_1 + \ldots + t_m v_m + c) = 0\}$$

be the complete relation set with generator $x \in \bar{F}$. The next lemma establishes a one-to-one correspondence between coordinate algebras of bounded algebraic sets $Y_F \subseteq V + c$ over F and coordinate algebras of algebraic sets $Y_k \subseteq k^m$ over k.

LEMMA 4.4. An F-algebra $C_F = \langle F, x\rangle$, where $x \in \bar{F}$, $x = t_1 v_1 + \ldots + t_m v_m + c$, and $t_1, \ldots, t_m \in \bar{k}$, is a realization of the coordinate algebra $\Gamma(Y_F)$ if and only if the k-subring $C_k = \langle k, t_1, \ldots, t_m\rangle$ is a realization of the coordinate ring $\Gamma(Y_k)$.

Proof. Repeating the argument used in the construction of $\mathrm{Rad}(Y_F) \leftrightarrow \mathrm{Rad}(Y_k)$, with field coefficients $\alpha_1, \ldots, \alpha_m \in k$ replaced by ring coefficients $t_1, \ldots, t_m \in \bar{k}$, we arrive at

$$R_F = \mathrm{Rad}(Y_F) \Leftrightarrow R_k = \mathrm{Rad}(Y_k).$$

Now the required result follows immediately. □

The next result — a consequence of Lemma 4.4 — gives a description of the coordinate algebra $\Gamma(Y_F)$ of a bounded algebraic set $Y_F \subseteq V + c$.

Proposition 4.5. Let $Y_F \subseteq V + c$ be a bounded algebraic set over an algebra F, and let $Y_k \subseteq k^m$ be its corresponding algebraic set over a field k. An F-algebra $C_F = \langle F, x\rangle$, $x \in \bar{F}$, is a realization of the coordinate algebra $\Gamma(Y_F)$ if and only if the following hold:

(1) the generator $x \in \bar{F}$ is representable as

$$x = t_1 v_1 + \ldots + t_m v_m + c, \quad t_1, \ldots, t_m \in \bar{k};$$

(2) the complete relation set R_k on coefficients $t_1, \ldots, t_m \in \bar{k}$ coincides with $\mathrm{Rad}(Y_k)$.
In particular, such algebras are all F-isomorphic.

The next proposition, which follows from the previous one, provides a description of the coordinate algebra $\Gamma(V + c)$ for an affine space $V + c$.

Proposition 4.6. Let $V + c \subset F$ be a finite-dimensional affine space. An F-algebra $C_F = \langle F, x\rangle$, $x \in \bar{F}$, is a realization of the coordinate algebra $\Gamma(V + c)$ if and only if the following hold:

(1) the generator $x \in \bar{F}$ is representable as

$$x = t_1 v_1 + \ldots + t_m v_m + c, \quad t_1, \ldots, t_m \in \bar{k};$$

(2) the coefficients $t_1, \ldots, t_m \in \bar{k}$ are such that:

(a) $\{(t_1^{(i)}, \ldots, t_m^{(i)}) \mid i \in I\} = k^m$ if k is finite;

(b) $\langle k, t_1, \ldots, t_m\rangle$ is a ring of polynomials in the variables $t_1, \ldots, t_m$ (or else $t_1, \ldots, t_m$ are algebraically independent over k) if k is infinite.

Proof. We need to show that for the case $Y_F = V + c$, conditions (2) in Propositions 4.5 and 4.6 are equivalent. By Corollary 1 to Theorem 4.3, we have $Y_k = k^m$. First we consider the case where $|k| = \infty$. Then $\mathrm{Rad}(k^m) = 0$, whence the result. If $|k| < \infty$ then $|k^m| < \infty$ and every subset $Y \subseteq k^m$ is algebraic; moreover, $\mathrm{Rad}(Y) < \mathrm{Rad}(k^m)$ if $Y < k^m$. □

General case. Translation between algebraic sets is possible also for the case of an arbitrary finite dimension $n > 1$. We repeat the above argument verbatim, the only deviation being that the notation will be more complex.

Let $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n)$ be a fixed n-parallelepiped. Similarly to the one-dimensional case, we define the correspondence $Y_F \leftrightarrow Y_k$ between bounded algebraic sets in $\mathbb{V}$ and algebraic sets in the affine space k^M as follows:

$$Y_F = \{(\alpha_1^1 v_1^1 + \ldots + \alpha_{m_1}^1 v_{m_1}^1 + c_1, \ldots, \alpha_1^n v_1^n + \ldots + \alpha_{m_n}^n v_{m_n}^n + c_n)\} \subseteq \mathbb{V} \leftrightarrow$$

$$Y_k = \{(\underbrace{\alpha_1^1, \ldots, \alpha_{m_1}^1}_{m_1}, \ldots, \underbrace{\alpha_1^n, \ldots, \alpha_{m_n}^n}_{m_n})\} \subseteq k^M$$

(here $M = m_1 + \ldots + m_n$).

An analog of Theorem 4.3 is also true. We have

THEOREM 4.7. Let $V_i + c_i$, $i = 1, \ldots, n$, be finite-dimensional affine subspaces of F (in dimensions $m_1, \ldots, m_n$, respectively), and let $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n) \subset F^n$ be an n-parallelepiped. Then $Y_F \leftrightarrow Y_k$ is a one-to-one correspondence between algebraic sets over F, lying in the n-parallelepiped $\mathbb{V}$, and algebraic sets over k, lying in the affine space k^M, where $M = m_1 + \ldots + m_n$.

The corollaries below also repeat those plugged in Theorem 4.3, and are proved similarly.

COROLLARY 5. The correspondence $Y_F \leftrightarrow Y_k$ associates an affine space k^M with the whole n-parallelepiped.

COROLLARY 6. There exists a one-to-one correspondence between radical ideals of a polynomial ring $k[y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^n, \ldots, y_{m_n}^n]$ and radicals of bounded algebraic sets over F lying in an n-parallelepiped $\mathbb{V}$. The latter may be characterized in terms of radical ideals of an algebra $F[x_1, \ldots, x_n]$, which contain Lie polynomials $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n)$ distinguishing affine spaces $V_1 + c_1, \ldots, V_n + c_n$, respectively.

COROLLARY 7. Let Y_F be any bounded algebraic set over F. Then there exists a finite system $S_0 \subset F[X]$ of equations such that $Y_F = V(S_0)$.

Proof. In fact, every bounded algebraic set Y_F is contained in some n-parallelepiped $\mathbb{V}$. Inside $\mathbb{V}$, we may define Y by finitely many equations in the same way as we did for the one-dimensional case. To these equations, then, we must add n equations $s_{m_1}(x_1 - c_1), \ldots, s_{m_n}(x_n - c_n)$ defining $\mathbb{V}$, which ultimately will yield the desired finite system S_0. $\square$

COROLLARY 8. If the ground field k is finite then any subset $M_F \subseteq \mathbb{V}$ of an n-parallelepiped $\mathbb{V}$ is algebraic over F.

COROLLARY 9. Let $Y_F \subseteq \mathbb{V}$ be a bounded algebraic set over an algebra F and let $Y_k \subseteq k^M$ be its corresponding algebraic set over a field k. An F-algebra $C_F = \langle F, x_1, \ldots, x_n \rangle$, $x_1, \ldots, x_n \in \bar{F}$, where

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \ldots, t_{m_i}^i \in \bar{k}, \quad i = 1, \ldots, n,$$

is a realization of the coordinate algebra $\Gamma(Y_F)$ if and only if a k-subring $C_k = \langle k, t_1^1, \ldots, t_{m_1}^1, \ldots, t_1^n, \ldots, t_{m_n}^n \rangle$ is a realization of the coordinate ring $\Gamma(Y_k)$.

As in the one-dimensional case, we have two propositions.

Proposition 4.8. Let $Y_F \subseteq \mathbb{V}$ be a bounded algebraic set over an algebra F and let $Y_k \subseteq k^M$ be its corresponding algebraic set over a field k. An F-algebra $C_F = \langle F, x_1, \ldots, x_n \rangle$, $x_1, \ldots, x_n \in \bar{F}$, is a realization of the coordinate algebra $\Gamma(Y_F)$ if and only if the following hold:

(1) the generators $x_1, \ldots, x_n \in \bar{F}$ are representable as

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \ldots, t_{m_i}^i \in \bar{k}, \quad i = 1, \ldots, n;$$

(2) the complete relation set R_k on coefficients $t_1^1,\dots,t_{m_1}^1,\dots,t_1^n,\dots,t_{m_n}^n \in \bar{k}$ coincides with $\mathrm{Rad}(Y_k)$. In particular, such algebras are all F-isomorphic.

Proposition 4.9. Let $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n) \subset F^n$ be an n-parallelepiped. An F-algebra $C_F = \langle F, x_1, \dots, x_n \rangle$, $x_1, \dots, x_n \in \bar{F}$, is a realization of the coordinate algebra $\Gamma(\mathbb{V})$ if and only if the following hold:

(1) the generators $x_1, \dots, x_n \in \bar{F}$ are representable as

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \dots, t_{m_i}^i \in \bar{k}, \quad i = 1, \dots, n;$$

(2) the coefficients $t_1^1,\dots,t_{m_1}^1,\dots,t_1^n,\dots,t_{m_n}^n \in \bar{k}$ are such that:

(a) $\{((t_1^1)^{(i)},\dots,(t_{m_1}^1)^{(i)},\dots,(t_1^n)^{(i)},\dots,(t_{m_n}^n)^{(i)}), \mid i \in I\} = k^M$ if k is finite;

(b) $\langle k, t_1^1,\dots,t_{m_1}^1,\dots,t_1^n,\dots,t_{m_n}^n \rangle$ is a ring of polynomials in the variables $t_1^1,\dots,t_{m_1}^1,\dots,t_1^n,\dots,t_{m_n}^n$ (or else $t_1^1,\dots,t_{m_1}^1,\dots,t_1^n,\dots,t_{m_n}^n$ are algebraically independent over k) if k is infinite.

What is the difference between the n- and one-dimensional cases for $m = M$, $M = m_1 + \ldots + m_n$? In either case bounded algebraic sets over F are in one-to-one correspondence with algebraic sets over k lying in k^M. For $m = M$, therefore, there are as many one-dimensional algebraic sets in the affine space $V + c$ as there are bounded algebraic sets inside the n-parallelepiped $\mathbb{V}$. The difference is that in the n-dimensional case, the construction of $Y_F \leftrightarrow Y_k$ proceeds by initially placing in k^M "partitions" between variables $y_1^1,\dots,y_{m_1}^1|\dots|y_1^n,\dots,y_{m_n}^n$, dividing the set of the variables into n parts. For algebraic sets, the difference between the *sorted* and the *monolith* varieties k^M is inessential. This stands out in treating the category $\mathcal{AS}(k)$ of algebraic sets over k as a whole, for the concept of *placing partitions* is not described in terms of morphisms of algebraic sets.

5. CLASSIFICATION OF BOUNDED ALGEBRAIC SETS

In this section we develop the idea of there being a correspondence between bounded algebraic set over an algebra F and algebraic sets over a field k, but not assuming, in this instance, that an n-parallelepiped is initially fixed. Our main goal is to describe objects of $\mathcal{B}AS(F)$, the category of all bounded algebraic sets over F, via translations into $\mathcal{AS}(k)$, the category of algebraic sets over k.

By definition, every bounded algebraic set Y_F over F is contained in some n-parallelepiped $\mathbb{V}$. Inside $\mathbb{V}$, in a one-to-one correspondence with Y_F is an algebraic set $Y_k \subseteq k^M$ over k. However, the correspondence $Y_F \to Y_k \bullet \mathbb{V}$, which associates a bounded set Y_F with a pair "n-parallelepiped $\mathbb{V}$ and algebraic set Y_k," appears inconvenient: such is not unique since $\mathbb{V}$ is not a sole n-parallelepiped in which Y_F is contained. We can get around this situation by defining a minimal n-parallelepiped of a bounded algebraic set Y_F.

Minimal n-parallelepiped. Let $Y_F \subset F^n$ be an arbitrary n-dimensional bounded algebraic set over F. Denote by $\mathbb{V}_Y$ the intersection of all n-parallelepipeds containing Y_F. Clearly,

(a) $\mathbb{V}_Y$ is also an n-parallelepiped;

(b) $\mathbb{V}_Y$ includes Y_F;

(c) $\mathbb{V}_Y$ is contained in any n-parallelepiped $\mathbb{V}$ such that $\mathbb{V} \supseteq Y_F$.

We call $\mathbb{V}_Y$ a *minimal n-parallelepiped* of a bounded algebraic set Y_F. For $\mathbb{V}_Y$, we use the same notation as was adopted for $\mathbb{V}$ in Sec. 2.

The definition of a minimal n-parallelepiped is not constructive. Here, therefore, we outline an extra procedure of finding a minimal n-parallelepiped, which is the simplest algorithm modulo the ring $\bar{k} = \prod\limits_{i \in I} k^{(i)}$.

Let $Y_F \subset F^n$ be any bounded algebraic set over F. Speaking of a bounded set being defined, we also mean that some n-parallelepiped $\mathbb{V} \supseteq Y_F$ is defined together with Y_F, as well as a realization of the coordinate algebra $\Gamma(Y_F)$ in $\bar{F}$, consistent with $\mathbb{V}$ (see Sec. 3), that is, $\Gamma(Y_F) = \langle F, x_1, \ldots, x_n \rangle$, where

$$x_i = t_1^i v_1^i + \ldots + t_{m_i}^i v_{m_i}^i + c_i, \quad t_1^i, \ldots, t_{m_i}^i \in \bar{k}, \quad i = 1, \ldots, n.$$

We show how to "diminish" the generators $x_1, \ldots, x_n \in \bar{F}$ for a minimal n-parallelepiped to be able to "accomodate" them. To do this, use will be made of the standard machinery for linear algebra.

By Lemma 3.3, the bounded algebra $B(\bar{F})$ is isomorphic to an algebra $F \otimes_k \bar{k}$, and hence $B(\bar{F})$ is a free module over the ring $\bar{k}$ with a basis consisting, for instance, of regular Hall words.

We take any element $x \in B(\bar{F})$ and represent it as

$$x = t_1 v_1 + \ldots + t_m v_m + c, \quad t_1, \ldots, t_m \in \bar{k}, \quad v_1, \ldots, v_m, c \in F. \tag{1}$$

We say that in correspondence with the representation of x is a finite-dimensional affine space $V + c$ in F, where $V = \mathrm{lin}_k\{v_1, \ldots, v_m\}$. Clearly, for x, representation (1) is not unique. In particular, we can reduce it, by decreasing the number of summands, in the following two cases:

(1) if $v_1, \ldots, v_m, c$ are linearly dependent over k;

(2) if $t_1, \ldots, t_m$ are affine dependent over k.

Using the reasoning standard for linear algebra, we see that for any element $x \in B(\bar{F})$, there are a representation of form (1), which we refer to as *irreducible*, and its corresponding affine space, which we conceive of as *minimal*, possessing the following properties:

(1) the elements $v_1, \ldots, v_m, c$ are linearly independent over k;

(2) the elements $t_1, \ldots, t_m$ are affine independent over k (moreover, an irreducible representation is unique up to permutation of summands in it, so its corresponding minimal space is also unique);

(3) no matter which reducible representation

$$x = t_1' v_1' + \ldots + t_n' v_n' + c', \quad t_1', \ldots, t_n' \in \bar{k}, \quad v_1', \ldots, v_n', c' \in F, \tag{2}$$

might be given, the number of summands in it is greater than is one in the irreducible representation, and doing away (in any order) with the linear dependence between $v_1', \ldots, v_n', c'$ and with the affine dependence between $t_1', \ldots, t_n'$, we are committed to be faced up to the unique irreducible representation desired;

(4) if $V' + c'$ is an affine space corresponding to an arbitrary representation (2) then $V' + c'$ contains $V + c$ as a minimal space.

Thus our algorithm is aimed at finding minimal spaces $V_1 + c_1, \ldots, V_n + c_n$ given some realization of the coordinate algebra $\Gamma(Y_F)$ in $\bar{F}$ such as $\Gamma(Y_F) = \langle F, x_1, \ldots, x_n \rangle$ for generators $x_1, \ldots, x_n \in B(\bar{F})$. Clearly, $\mathbb{V} = (V_1 + c_1) \times \ldots \times (V_n + c_n)$ will be a minimal n-parallelepiped for the bounded set Y_F. Since our minimal n-parallelepiped is unique, its construction does not depend on the starting realization of the coordinate algebra $\Gamma(Y_F)$.

Let $Y_F \subset F^n$ be a bounded algebraic set over an algebra F. As noted, the non-uniqueness of $Y_F \to Y_k \bullet \mathbb{V}$ is associated with ambiguities in defining an n-parallelepiped $\mathbb{V} \supseteq Y_F$. As $\mathbb{V}$ we now take a minimal n-parallelepiped $\mathbb{V}_Y$ of Y_F and define a correspondence such as $Y_F \to Y_k \bullet \mathbb{V}_Y$, where Y_k is an algebraic set over a field k corresponding to Y_F treated as a subset of the n-parallelepiped $\mathbb{V}_Y$.

The fact that $Y_F \leftrightarrow Y_k$ is realized inside the minimal n-parallelepiped $\mathbb{V}_Y$ imposes on $Y_k \subseteq k^M$ the following maximality condition.

Definition. Let $Y_k \subseteq k^M$ be an algebraic set over a field k. We say that Y_k *satisfies the maximality condition* relative to the partition $M = m_1+\ldots+m_n$ if the radical $\mathrm{Rad}(Y_k) \subset k[y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^n, \ldots, y_{m_n}^n]$ lacks in non-trivial equations such as

$$\begin{gathered} \beta_1^1 y_1^1 + \ldots + \beta_{m_1}^1 y_{m_1}^1 + \gamma_1 = 0, \quad \beta_1^1, \ldots, \beta_{m_1}^1, \quad \gamma_1 \in k, \\ \ldots \\ \beta_1^n y_1^n + \ldots + \beta_{m_n}^n y_{m_n}^n + \gamma_n = 0, \quad \beta_1^n, \ldots, \beta_{m_n}^n, \quad \gamma_n \in k. \end{gathered} \tag{3}$$

In other words, if Y_k is written in the form of the Cartesian product $Y_k = Y^1 \times \ldots \times Y^n$, where $Y^1 \subset k^{m_1}, \ldots, Y^n \subset k^{m_n}$, then no one of the sets Y^i is contained in an affine space of dimension lesser than m_i, $i = 1, \ldots, n$.

LEMMA 5.1. Let $Y_F \subset F^n$ be a bounded algebraic set over F, $\mathbb{V}_Y$ be a minimal n-parallelepiped for Y_F, and $Y_k \subseteq k^M$ be an algebraic set over k corresponding to Y_F inside $\mathbb{V}_Y$. Then Y_k satisfies the maximality condition relative to the partition $M = m_1 + \ldots + m_n$.

The **proof** follows the line of argument used in describing the algorithm for finding a minimal n-parallelepiped. Let, to the contrary,

$$\beta_1 y_1^1 + \ldots + \beta_{m_1} y_{m_1}^1 + \gamma 1 \in \mathrm{Rad}(Y_k), \quad \beta_1 \neq 0.$$

For any point $(\alpha_1^1, \ldots, \alpha_{m_1}^1, \ldots, \alpha_1^n, \ldots, \alpha_{m_n}^n) \in Y_k$, we then have

$$\alpha_1^1 = -(\beta_2 : \beta_2) \cdot \alpha_2^1 - \ldots - (\beta_{m_1} : \beta_1) \cdot \alpha_{m_1}^1 - (\gamma_1 : \beta_1).$$

Consequently, for any point $(b_1, \ldots, b_n) \in Y_F$, it is true that

$$b_1 = \alpha_1^1 v_1^1 + \ldots + \alpha_{m_1}^1 v_{m_1}^1 + c_1 =$$

$$\alpha_2^1 \cdot (v_2^1 - (\beta_2 : \beta_1) \cdot v_1^1) + \ldots + \alpha_{m_1}^1 \cdot (v_{m_1}^1 - (\beta_{m_1} : \beta_1) \cdot v_1^1) + (c_1 - (\gamma_1 : \beta_1) \cdot v_1^1),$$

which is a contradiction with the minimality of an n-parallelepiped $\mathbb{V}_Y$. □

If we inverse the argument used in the proof of Lemma 5.1 we arrive at the following:

LEMMA 5.2. Let $\mathbb{V}$ be an n-parallelepiped, and let $Y_k \subseteq k^M$ and $Y_F \subset F^n$ be mutually corresponding algebraic sets inside $\mathbb{V}$. If Y_k satisfies the maximality condition relative to the partition $M = m_1 + \ldots + m_n$ then $\mathbb{V}$ is a minimal n-parallelepiped for Y_F.

Lemmas 5.1 and 5.2 imply that $Y_F \to Y_k \bullet \mathbb{V}_Y$ is invertible and is one-to-one.

Denote by $\mathcal{A}\!f\!f(F)$ the subcategory in $\mathcal{B}AS(F)$ whose objects are all possible n-parallelepipeds, with $n \in \mathbb{N}$. Define the category $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$, taking as its objects the set of all consistent pairs of the form $Y_k \bullet \mathbb{V}_Y$, where

(1) $\mathbb{V}_Y$ is an n-parallelepiped (with ranks $m_1, \ldots, m_n$ and with bases of its generating affine spaces defined as in Sec. 2);

(2) Y_k ($\subseteq k^M$) is an algebraic set over k satisfying the maximality condition relative to $M = m_1+\ldots+m_n$.

Morphisms of the category $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$ are naturally generated by those in $\mathcal{A}S(k)$ and in $\mathcal{A}\!f\!f(F)$.

Lemmas 5.1 and 5.2 imply the following:

THEOREM 5.3. Objects of the category $\mathcal{B}AS(F)$ are in one-to-one correspondence with objects in $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$.

COROLLARY. If the ground field k is finite then bounded algebraic sets over F are exactly all possible finite pointsets.

Despite the fact that the objects in $\mathcal{B}AS(F)$ are in one-to-one correspondence with those in $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$, the categories themselves are not isomorphic. The reason is that $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$, by definition, enjoys more morphisms than $\mathcal{B}AS(F)$. The categories $\mathcal{B}AS(F)$ and $\mathcal{A}S(k) \bullet \mathcal{A}\!f\!f(F)$ will be isomorphic if we redefine morphisms in one of them, or in both.

6. ALGEBRAIC GEOMETRY OVER A FREE LIE ALGEBRA IN DIMENSION 1

Here, we clarify which algebraic sets $Y \subseteq F$ in dimension 1 are unbounded. By Theorem 1.3, the coordinate algebra $\Gamma(Y)$ has the following realization in the Cartesian product $\bar{F} = \prod\limits_{i \in I} F^{(i)}$:

$$\Gamma(Y) = \langle F, x \rangle, \quad x \in \bar{F}.$$

If the generator $x \in \bar{F}$ belongs to a bounded subalgebra $B(\bar{F})$ then $\Gamma(Y)$ is a bounded coordinate algebra and Y is a bounded algebraic set. Therefore we assume that $x \notin B(\bar{F})$.

LEMMA 6.1. If $x \notin B(\bar{F})$ then the coordinate algebra $\Gamma(Y)$ is F-isomorphic to a free Lie algebra with generators $a_1, \ldots, a_r, x$, where $a_1, \ldots, a_r$ are free generators for F.

Proof. We claim that $\Gamma(Y) \cong_F F[x]$. Assume, to the contrary, that there exists a non-zero Lie polynomial $f(x) \in F[x]$ with roots in arbitrarily large degrees. We find an element $x_0 \in F$, the root of $f(x)$, such that the degree of x_0 is greater than is one of $f(x)$ treated as an element of the free Lie algebra $F[x]$. The equality $f(x_0) = 0$ implies $f(x) \in \mathrm{id}\langle x - x_0 \rangle$, where $\mathrm{id}\langle x - x_0 \rangle$ is an ideal of $F[x]$ generated by $(x - x_0)$. By the choice of $x_0 \in F$, the polynomial $f(x)$ cannot belong to $\mathrm{id}\langle x - x_0 \rangle$ since the degree of any non-zero element of $\mathrm{id}\langle x - x_0 \rangle$ is not less than is one of its generating element $(x - x_0)$ (see [8]). $\square$

COROLLARY. For the case where $x \notin B(\bar{F})$, the whole algebra F is an algebraic set $Y \subseteq F$ corresponding to a coordinate algebra $\langle F, x \rangle$.

Thus the following theorem holds.

THEOREM 6.2. Any algebraic set Y in dimension 1 ($Y \subseteq F$) over a free Lie algebra F is one of the following:

(a) a bounded set, or

(b) the whole algebra F.

Recall that for the case where k is a finite field, bounded algebraic sets are all possible finite pointsets.

CONCLUSION

As noted in the Introduction, all the results of the present paper can be extended to the case where F is a free anticommutative algebra over a field k. Below, we give a list of the properties of a free Lie algebra which have been used in our account.

Property 1. Let $v \in F$ be a non-zero element. Then a solution for $s(x) = x \circ v = 0$ is exactly a one-dimensional space spanned by the vector v.

This implies that n-parallelepipeds are algebraic sets over F.

Property 2. Let $a_1, \ldots, a_r$ be free generators for F. Then F has a linear basis consisting of words composed of letters $a_1, \ldots, a_r$. For every element $a \in F$, we define the degree n relative to $a_1, \ldots, a_r$. For every natural n, the set of basis elements whose degrees do not exceed n is finite. For a free Lie algebra, such a basis can be exemplified by the Hall basis.

Property 3. Let $a, b \in F$, $a \circ b \neq 0$. Then the degree of a product $a \circ b$ is greater than the degrees of elements a and b. If $a, b_1, \ldots, b_n \in F$ are non-zero elements, and the degree of a is greater than the degrees of b_i, $i = 1, \ldots, n$, then $a \circ b_1 \circ \ldots \circ b_n \neq 0$.

This property is used in translating equations over a field k into equations over an algebra F.

Property 4. Let $a \in F$ be a non-zero element, and let $\mathrm{id}\langle a \rangle$ be an ideal of F generated by a. Then the degree of any non-zero element of $\mathrm{id}\langle a \rangle$ is not less than is one of a (see [8]).

Property 4 was appealed to in the proof of Lemma 6.1, using which we have obtained a description of all algebraic sets over F in dimension 1 ($n = 1$).

In the present paper, we did not attempt to find all algebras satisfying Properties 1-4. Still it is worth observing that a free anticommutative algebra satisfies these, and hence all of the results presented above hold true for it, too.

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