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Algebraic Geometry over Groups II. Logical Foundations

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The object of this paper, which is the second in a series of three, is to lay the logical foundations of the algebraic geometry over groups. Exploiting links between the algebraic geometry over groups and model theory we solve two problems on geometrical equivalence of groups which are due to B. Plotkin.

CONTENTS

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1. INTRODUCTION

In [BMR1] we introduced and studied some basic notions of algebraic geometry over groups. It turns out that these notions have interesting and deep connections with logic and model theory. On the one hand, this brings powerful methods of model theory into algebraic geometry over groups and shows how to develop this type of geometry in the more general framework of model theory. On the other hand, these connections yield new applications of certain methods of universal algebra, and lead to some new ideas in that area. To explain this we need to recall a few definitions. We refer to [BMR1] for algebraic geometry over groups, to [Mal2] for universal algebra, and to [CK] for model theory.

The standard language of group theory, which we denote by L, consists of a symbol for multiplication $\cdot$, a symbol for inversion $^{-1}$, and a symbol for the identity 1. For the purpose of algebraic geometry over a given fixed group G one has to consider the category of G-groups, i.e., groups which contain the group G as a distinguished subgroup. In this category morphisms are G-homomorphisms, i.e., homomorphisms which are identical on the distinguished subgroup G. Subgroups are G-subgroups, i.e., subgroups containing the distinguished subgroup G, etc. (see [BMR1] for details). To deal with G-groups, we enlarge the language L by all non-trivial elements from G as new constants and denote this new language by L_G. It is understood that all the groups that we consider in this paper are G-groups, and the language we consider them in is L_G (unless stated otherwise). Since $L_G = L$ for $G = 1$, the case of ordinary groups is covered as well.

With a given G-group H one can associate several model-theoretic classes of G-groups: the *variety* $\mathrm{var}(H)$ consisting of all G-groups which satisfy every identity in the language L_G that holds in H, the *quasivariety* qvar(H) consisting of all G-groups which satisfy every quasi-identity in the language L_G that holds in H, and the universal closure ucl(H) consisting of all G-groups which satisfy every universal sentence in the language L_G that holds in H. It will be convenient to have a purely algebraic characterization of the logical classes above. To this end, for a class of groups $\mathcal{K}$ we denote by $S(\mathcal{K})$, $P(\mathcal{K})$, $P_u(\mathcal{K})$, and $H(\mathcal{K})$ the classes of all groups isomorphic to subgroups, unrestricted Cartesian products, ultraproducts, and homomorphic images of groups from $\mathcal{K}$, respectively. It is known that $\mathrm{var}(H) = HSP(H)$ [Bir], $\mathrm{qvar}(H) = SPP_u(H)$ [GL], and $\mathrm{ucl}(H) = SP_u(H)$ [Mal2]. We need one more class, the *prevariety* $\mathrm{pvar}(H) = SP(H)$ generated by H, which, in general, is not axiomatizable. Clearly,

$$\mathrm{pvar}(H), \mathrm{ucl}(H) \subseteq \mathrm{qvar}(H) \subseteq \mathrm{var}(H).$$

These classes are the key players in what follows and we discuss them in detail in Section 1.

Let X be a set and let $F(X)$ be a free group with basis X. The free product $G[X] = G * F(X)$ is a free object in the category of G-groups, we call it a *free G-group* on X. A subset $S \subset G[X]$ defines a system of equations $S(X) = 1$ over G (in variables from X and with constants from G). Fix an arbitrary G-group H. The solution set of $S(X) = 1$ in $H^{|X|}$ is called the *algebraic set* $V_H(S)$ defined by S. One and the same algebraic set can be defined by different systems of equations, but for a given subset S there exists a unique maximal subset $\mathrm{Rad}(S)$ of $G[X]$ (which is called the *radical* of S with respect to H) with the property $V_H(S) = V_H(\mathrm{Rad}(S))$. In fact, the radical $\mathrm{Rad}(S)$ is a normal subgroup of $G[X]$ and the quotient group

$$G_{R(S)} = G[X]/\,\mathrm{Rad}(S)$$

is called the *coordinate group* of the algebraic set $V_H(X)$. If $V_H(S)$ is non-empty (the only interesting case) then $G_{R(S)}$ is a G-group. Furthermore, if the algebraic sets $V_H(S)$ and $V_H(T)$ defined by two systems $S(X) = 1$ and $T(X) = 1$ are non-empty, then their coordinate groups $G_{R(S)}$ and $G_{R(T)}$ are G-isomorphic if and only if $V_H(S)$ and $V_H(T)$ are equivalent [BMR1]. This reduces the study of algebraic sets to their coordinate groups and radicals.

For a class of groups $\mathcal{K}$, we denote by $\mathcal{K}_\omega$ the class of all finitely generated G-groups from $\mathcal{K}$. This notation will be used throughout the paper. The following result is the first link between algebraic geometry over groups and universal algebra.

PROPOSITION A. *Let H be a G-group. Then every group from* $\mathrm{pvar}(H)_\omega$ *is the coordinate group of an algebraic set over H, and conversely, every such coordinate group belongs to* $\mathrm{pvar}(H)_\omega$.

Characterizing radicals is a difficult and important problem in every version of algebraic geometry. Hilbert's famous Nullstellensatz gives a description of radicals in the classical case. In view of that, we will refer to the *generalized Nullstellensatz* as to any "reasonable" description of radicals in all their variations.

It is easy to see that the radical $\mathrm{Rad}(S)$ is equal to the intersection of the kernels of all G-homomorphisms from $G[X]$ into H that contain S. This is a particular case of a very general definition of a radical with respect to an arbitrary class of groups $\mathcal{K}$ which is closed under Cartesian products and subgroups. Namely, a normal subgroup N of a group A is called *$\mathcal{K}$-radical* if $A/N \in \mathcal{K}$. Similarly, the $\mathcal{K}$-radical of a subset $Y \subseteq A$ is the intersection of all $\mathcal{K}$-radical subgroups of A containing Y. For a given set $S \subseteq G[X]$ we now define three new types of radicals of S, namely, the $\mathcal{K}$-radicals in the above sense when $\mathcal{K}$ is $\mathrm{pvar}(H)$, $\mathrm{qvar}(H)$, and $\mathrm{var}(H)$, respectively. We denote them by $\mathrm{Rad}_p(S)$, $\mathrm{Rad}_q(S)$, and $Rad_v(S)$. It turns out that $\mathrm{Rad}(S) = \mathrm{Rad}_p(S)$ for any G-group H and any set $S \subset G[X]$ (Lemma 3, Section 2.2). Thus

$$\mathrm{Rad}(S) = \mathrm{Rad}_p(S) \supseteq \mathrm{Rad}_q(S) \supseteq \mathrm{Rad}_v(S).$$

As we have mentioned above, the class $\mathrm{pvar}(H)$ is not always axiomatizable in first-order logic, and therefore, in general, one cannot expect to be able to describe the radical $\mathrm{Rad}_p(S)$ in a "reasonable" way. On the contrary, the classes $qvar(H)$ and $var(H)$ are axiomatizable (by quasi-identities and identities), and hence the radicals $\mathrm{Rad}_q(S)$ and $\mathrm{Rad}_v(S)$ consist of all logical consequences of the system $S = 1$ in the quasi-equational and equational logics, respectively (see Section 2.2). Now we can formulate the Nullstellensatz in a more precise form. We say that a set $S \subseteq G[X]$ *satisfies the Nullstellensatz over H* if $\mathrm{Rad}_p(S) = \mathrm{Rad}_v(S)$. The following argument shows that in many cases this form of the Nullstellensatz gives the best possible description of the radical. Indeed, if G is a non-Abelian hyperbolic group (for example, a non-Abelian free group) then the free G-group $G[X]$ is a subgroup of a Cartesian power of G [BMR1], and hence it belongs to the variety $\mathrm{var}(H)$. Therefore, every normal subgroup of $G[X]$ that intersects G trivially is $\mathrm{var}(H)$-radical. This shows that if a system $S(X) = 1$ has a solution in H then $\mathrm{Rad}_v(S) = \langle S \rangle^{G[X]}$ is the normal closure of S in $G[X]$. This form of the Nullstellensatz is due to Rips, and it was discussed in detail in [BMR1].

One can find in [KM] a description of the radicals of quadratic equations over a free non-Abelian group F. It turns out that almost all quadratic equations satisfy the Nullstellensatz over F, but there are some for which the Nullstellensatz does not hold. It seems that systems satisfying this, the strongest form of the Nullstellensatz over F, are scarce.

The following question arises naturally: what minimal conditions should be imposed on the class $\mathrm{pvar}(H)$ to allow a description of the radical $\mathrm{Rad}_p(S)$ as a set of logical consequences of the system $S = 1$? For example, one such condition on $\mathrm{pvar}(H)$ is axiomatizibility. Malcev proved that $\mathrm{pvar}(H)$ is axiomatizable if and only if $\mathrm{pvar}(H) = \mathrm{qvar}(H)$ [Mal1]. To this end, we say that a set $S \subseteq G[X]$ satisfies the *generalized Nullstellensatz* over H if $\mathrm{Rad}_p(S) = \mathrm{Rad}_q(S)$. Thus, the generalized Nullstellensatz is closely related (but not equivalent!) to the following well-known problem of Malcev: for which classes of groups $\mathcal{K}$ does the equality $\mathrm{pvar}(\mathcal{K}) = \mathrm{qvar}(\mathcal{K})$ hold? A complete solution of this problem has recently been given by Gorbunov [Gor]. Recall that an equation $f = 1$ is a consequence of a system $S(X) = 1$ over a class of groups $\mathcal{K}$ if every solution of $S = 1$ is also a solution of $f = 1$ in every group from $\mathcal{K}$. Following [Gor], we say that a class of groups $\mathcal{K}$ is *q-compact* if for an arbitrary set X and an arbitrary system of equations $S(X) = 1$ any consequence $f(X) = 1$ of the system $S(X) = 1$ over $\mathcal{K}$ is a consequence of some finite subsystem of $S = 1$ over $\mathcal{K}$. Observe,that in the definition above the set X is allowed to be infinite. Gorbunov proved that for a class of groups $\mathcal{K}$ the prevariety $\mathrm{pvar}(\mathcal{K})$ is a quasivariety if and only if $\mathcal{K}$ is q-compact.

Unfortunately, q-compactness is a very restrictive condition on the class $\mathcal{K}$. Indeed, if a group H contains an infinite cyclic subgroup and does not contain any non-trivial divisible Abelian subgroups (for example, if H is an infinite hyperbolic group) then the group H, i.e., the class $\{H\}$, is not q-compact (Proposition 6).

Fortunately, if in the system of equations $S(X) = 1$ the set X is finite (this is always the case in the algebraic geometry over H) the radical $\mathrm{Rad}_p(S)$ depends only on *finitely generated* G-groups from $\mathrm{pvar}(H)$. This shows that for the purposes of algebraic geometry it is more natural to consider the following variation of the Malcev problem (the *restricted Malcev* problem): describe the classes $\mathcal{K}$ for which $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$. It turns out that the restricted Malcev problem has a similar solution to the original one. Namely, we say that a class $\mathcal{K}$ is *q_ω-compact* if it satisfies the definition of q-compactness above with the extra requirement on the set X to be *finite*. Clearly, every q-compact class is also q_ω-compact. The following theorem solves the restricted Malcev problem.

THEOREM B1. *Let G be a group and let $\mathcal{K}$ be a class of G-groups. Then* $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$ *if and only if $\mathcal{K}$ is q_ω-compact.*

The main corollary of this result is that over a q_ω-compact group H every system of equations in finitely many variables satisfies the generalized Nullstellensatz. It is worthwhile to note here that there are many q_ω-compact groups. For example, every G-equationally Noetherian group is q_ω-compact. Recall that a group H is termed *G-equationally Noethe-*

rian if every system of equations in finitely many variables and with coefficients from G is equivalent over H to a finite subsystem of it. We refer to [BMRom, BMR1] for a detailed discussion of equationally Noetherian groups. Here mention only that every Abelian or linear group (over a unitary Noetherian commutative ring) is equationally Noetherian, and hence q_ω-compact. It follows that every linear torsion-free hyperbolic group (for instance, a free group) is q_ω-compact but is not q-compact.

Although, in general, $\mathrm{qvar}(\mathcal{K}) \neq \mathrm{pvar}(\mathcal{K})$ for a q_ω-compact class $\mathcal{K}$, groups from $\mathrm{qvar}(\mathcal{K})$ are easy to describe. To this end, we say that a G-group H is *G-separated* by a class of groups $\mathcal{K}$ if for any non-trivial element $h \in H$ there exists a group $A \in \mathcal{K}$ and a G-homomorphism $\phi: H \to A$ such that $\phi(h) \neq 1$. Denote by $\mathrm{Sep}(\mathcal{K})$ the class of all G-groups which are G-separated by $\mathcal{K}$, and by $L\mathcal{K}$ the class of all *locally* $\mathcal{K}$-groups, i.e., groups in which every finitely generated G-subgroup belongs to $\mathcal{K}$. It follows from Theorem B1 by a standard model-theoretic argument that a class $\mathcal{K}$ is q_ω-compact if and only if $\mathrm{qvar}(\mathcal{K}) = L\,\mathrm{Sep}(\mathcal{K})$. We discuss all these results in detail in Sections 3.1 and 3.2.

In Section 3.3 we introduce a notion of *u-compactness*, which is a variation of the compactness theorem for universal formulas. If a class $\mathcal{K}$ is u-compact then its universal closure $\mathrm{ucl}(\mathcal{K})$ admits a description similar to the one mentioned above. More precisely, recall that a G-group H is *G-discriminated* by a class of groups $\mathcal{K}$ if for any finite subset of non-trivial elements $H_0 \subset H$ there exists a group $A \in \mathcal{K}$ and a G-homomorphism $\phi: H \to A$ such that $\phi(h) \neq 1$ for every $h \in H_0$. Denote by $\mathrm{Dis}(\mathcal{K})$ the class of all G-groups that are G-discriminated by $\mathcal{K}$. The following result gives the desired description.

THEOREM B2. *Let $\mathcal{K}$ be a class of G-groups. Then $\mathcal{K}$ is u-compact if and only if* $\mathrm{ucl}(\mathcal{K}) = L\,\mathrm{Dis}(\mathcal{K})$.

Again, it is worthwhile to note that there are many u-compact groups. For example, every G-equationally Noetherian group is u-compact. As a corollary of the theorem above we have a generalization of the criterion of G-universal equivalence from [BMR1].

COROLLARY B3. *Let H and K be u-compact G-groups. Then H is G-universally equivalent to K if and only if H is locally G-discriminated by K and K is locally G-discriminated by H.*

Notice that u-compactness is only required in one direction of the above equivalence. If H and K locally G-discriminate each other, then they are G-universally equivalent without any additional assumptions (see the proof in [BMR1]).

In [Plot1] Plotkin introduced an important concept of geometrically equivalent algebraic structures. Here we discuss only geometrically equivalent groups, but all the results extend to arbitrary algebraic structures. Two G-groups H and K are called *geometrically equivalent* if for any finite set X and for any subset $S(X) \subset G[X]$ the system of equations $S(X) = 1$ has precisely the same radicals with respect to $\text{pvar}(H)$ and $\text{pvar}(K)$, i.e., $\text{Rad}_H(S) = \text{Rad}_K(S)$. The point here is that geometrically equivalent groups have similar algebraic geometries which allows one to try and classify groups with respect to their geometric properties. Indeed, if H and K are geometrically equivalent then for any $S \subseteq G[X]$ the algebraic sets $V_H(S)$ and $V_K(S)$ have exactly the same coordinate groups,

$$G[X]/\operatorname{Rad}_H(S) = G[X]/\operatorname{Rad}_K(S);$$

i.e., the categories of coordinate groups (defined by equations with coefficients from G) over H and over K coincide. This implies that the categories of algebraic sets over H and over K (defined by equations with coefficients from G) are equivalent (see [BMR1, Plot1] for details).

The following observation is due to Plotkin [Plot1]. If the groups H and K are geometrically equivalent then $\text{qvar}(H) = \text{qvar}(K)$. This led him to formulate the following problem [Plot1, Problem 6].

(P1) Let H and K be G-groups and let $\text{qvar}(H) = \text{qvar}(K)$. Does this imply that H and K are geometrically equivalent?

The condition $\text{qvar}(H) = \text{qvar}(K)$ means that the groups H and K satisfy precisely the same quasi-identities in the language L_G. In his other problem [Plot1, Problem 7]) Plotkin strengthened this condition, requesting the groups H and K to be elementarily equivalent. Recall that two G-groups are *elementarily equivalent* if they satisfy exactly the same sentences in the language L_G.

(P2) Let H and K be two elementarily equivalent G-groups. Does this imply that H and K are geometrically equivalent?

The following result gives negative solutions of problems (P1) and (P2).

THEOREM C1. *Let H be a G-group which is not q_ω-compact. Then there exists an ultrapower K of H such that H and K are not geometrically equivalent.*

Since an ultrapower K of H is elementarily equivalent to H the theorem above shows that if H is not q_ω-compact then there exists a group K for which Plotkin's questions above have negative answers. One can even find finitely generated G-groups H and K which provide a negative answer to problems (P1) and (P2) (see Corollary 9 in Section 4.2).

Now we turn to the question of what conditions the equality $\text{qvar}(H) = \text{qvar}(K)$ imposes onto groups H and K in terms of radicals. To this end, G-groups H and K are termed *ω-geometrically equivalent* if for any finite set X and any finite subset $S \subseteq G[X]$ the equality $\mathrm{Rad}_H(S) = \mathrm{Rad}_K(S)$ holds. Obviously, geometrically equivalent groups are also ω-geometrically equivalent.

THEOREM C2. *Let H and K be G-groups. Then* $\mathrm{qvar}(H) = \mathrm{qvar}(K)$ *if and only if H and K are ω-geometrically equivalent.*

On the other hand, problems (P1) and (P2) have affirmative solutions for q_ω-compact groups.

THEOREM C3. *Let H and K be q_ω-compact G-groups. Then H and K are geometrically equivalent if and only if* $\mathrm{qvar}(H) = \mathrm{qvar}(K)$.

In fact, there is an interesting characterization of q_ω-compact groups in terms of geometrical equivalence.

THEOREM C4. *Let G be a q_ω-compact group. Then any two groups from* $\mathrm{qvar}(G)$ *are geometrically equivalent. And vice versa, if any two groups from* $\mathrm{qvar}(G)$ *are geometrically equivalent then G is q_ω-compact.*

As we have already mentioned Abelian groups and linear groups are q_ω-compact. Therefore, if H and K are Abelian or linear groups (in particular, finitely generated nilpotent groups) then they are geometrically equivalent if and only if $\mathrm{qvar}(H) = \mathrm{qvar}(K)$. All these results are discussed in Section 4.

In Section 5 we characterize G-equationally Noetherian groups in terms of the quasivarieties generated by them. It turns out that a G-group H is G-equationally Noetherian if and only if the quasivariety $\mathrm{qvar}(H)$ generated by H is Noetherian. Namely, the following theorem holds.

THEOREM D1. *Let H be a G-group. Then the following conditions are equivalent*:

(1) *H is G-equationally Noetherian*;

(2) *every finitely generated G-group in the quasivariety* $\mathrm{qvar}(H)$ *is finitely presented in* $\mathrm{qvar}(H)$;

(3) *every free G-group $F_H(X)$ of finite rank in* $\mathrm{qvar}(H)$ *satisfies the ascending chain condition on* $\mathrm{qvar}(H)$*-radical subgroups*;

(4) *every free G-group $G[X]$ of finite rank satisfies the ascending chain condition on* $\mathrm{qvar}(H)$*-radical subgroups.*

Moreover, if H is G-equationally Noetherian then every G-group in $\mathrm{qvar}(H)$ *is G-equationally Noetherian.*

Notice that the classification of Noetherian quasivarieties is one of the older problems in universal algebra.

In Section 6 we study the coordinate groups of algebraic sets in some particular situations. Let H be a G-group. It is not hard to see that the coordinate group of the algebraic set H^n is the free group F_H of rank n in the variety var(H) (Proposition 8). In general, this group may have a rather complicated structure. In Section 6.1 we discuss the structure of F_H in the case when H is torsion-free hyperbolic, or Abelian, or free in some nilpotent variety.

In Section 6.2 we describe coordinate groups over a fixed Abelian group A. To explain this we need few definitions. The *period* of an Abelian group A is the smallest positive integer m, if it exists, such that $mA = 0$, and ∞ otherwise. Denote by $e(A)$ the period of A and by $e_p(A)$ the period of the p-primary component of A for a prime number p.

THEOREM D2. *Let A be an Abelian group. Then a finitely generated A-group B is a coordinate group of some algebraic set over A if and only if B satisfies the conditions*:

(1) $B \simeq A \oplus C$, *where C is a finitely generated Abelian group*;

(2) $e(A) = e(B)$ *and* $e_p(A) = e_p(B)$ *for every prime number p.*

Since every Abelian group is q_ω-compact, the finitely generated A-groups in pvar(A) and qvar(A) are exactly the same. Therefore the generalized Nullstellensatz holds for every system of equations $S = 1$ over A. This allows one to describe the radical $\mathrm{Rad}_A(S)$ as the set of all consequences of S with respect to the set of quasi-identities $Q(A)$ which hold in A. In Theorem 1 we describe the set $Q(A)$ for an arbitrary Abelian group A. This result is interesting in its own right. In [Fed] Fedoseeva gives a description of all algebraic sets over A up to equivalence. Based on our description of coordinate groups over A we give another, simpler, proof of the results from [Fed].

Recall that for a G-group H and a positive integer n we term a subset of H^n to be *closed* if it is the intersection of an arbitrary number of finite unions of algebraic sets in H^n. This defines a topology on H^n, called the *Zariski* topology. The Zariski topology over H is *Noetherian* (i.e., it satisfies the descending chain condition on closed subsets) if and only if the group H is G-equationally Noetherian [BMR1]. In this event every closed set is a finite union of algebraic sets. As is the custom in topology, a closed set is called *irreducible* if it is not a union of two proper closed subsets. If H is G-equationally Noetherian then every topology subset Y of H^n closed in the Zariski is a finite union of irreducible algebraic sets (the *irreducible components* of Y), each of which is uniquely determined by Y. The following result from Section 7.1 gives a model-theoretic characterization of coordinate groups of irreducible algebraic sets.

THEOREM E1. *Let H be a G-equationally Noetherian group. Then a finitely generated G-group K is the coordinate group of an irreducible al-*

gebraic set over H if and only if $\mathrm{ucl}(K) = \mathrm{ucl}(H)$; *i.e., K is G-universally equivalent to H.*

In Section 7.2 we describe a simple set of axioms which defines the class $\mathrm{ucl}(H)$ inside the quasivariety $\mathrm{qvar}(H)$. Here we discuss just one particular case of this result—when the group H is a *CSA group.* Recall that a group H is called CSA if every maximal Abelian subgroup M of H is malnormal, i.e., $h^{-1}Mh \cap M = 1$ for any $h \in H - M$. We refer to [MR2] for various facts about CSA groups. However, it is worthwhile to mention that every torsion-free hyperbolic group is CSA and every CSA group is *commutative transitive*, i.e., satisfies the following axiom:

$$CT = \forall x\, \forall y\, \forall z \big(x \neq 1 \wedge y \neq 1 \wedge z \neq 1 \wedge [x, y] = 1 \wedge [x, z] = 1 \rightarrow [y, z] = 1\big).$$

THEOREM E2. *Let H be a non-Abelian G-equationally Noetherian CSA G-group. Then*:

(1) *A non-Abelian finitely generated G-group K from* $\mathrm{qvar}(H)$ *is a coordinate group of some irreducible algebraic set over H if and only if K is CSA*;

(2) *A non-Abelian G-group K belongs to* $\mathrm{ucl}(H)$ *if and only if K satisfies the set of axioms $Q(H) \cup \{CT\}$ (here $Q(H)$ is the set of all quasi-identities which hold on H).*

If G is non-Abelian then CSA G-groups form a proper subclass of the much larger class of so-called G-domains. It turns out that a result similar to Theorem E2 holds for an arbitrary G-domain H (see Section 7.2).

Section 7.3 contains a description of the coordinate groups of irreducible algebraic sets over a fixed Abelian group A. To formulate this result we denote by $\alpha_{p^k}(A)$ the number of elements of order p^k in A (we allow $\alpha_{p^k}(A) = \infty$), where k is a positive integer and p is a prime number.

THEOREM E3. *Let A be an Abelian group and let B be a finitely generated A-group. Then B is a coordinate group of some irreducible algebraic set over A if and only if the following conditions hold*:

(1) *$B \simeq A \oplus C$ for some finitely generated Abelian group C*;

(2) *$e(A) = e(B)$ and $e_p(A) = e_p(B)$ for each prime number p*;

(3) *$\alpha_{p^k}(A) = \alpha_{p^k}(B)$ for each prime number p and positive integer k.*

Based on this theorem we give a set of axioms for the universal closure $\mathrm{ucl}(A)$ of A (Theorem 2).

The classical notion of an algebraic linear group over a field k plays an important part in group theory. In Section 8 we introduce a new class of groups, which we call *algebraic groups* over a given group H, and study their connected components in the Zariski topology. We do not discuss these results here in full generality. Instead we mention just one particular corollary for the group H itself. It turns out that if H is G-equationally Noetherian then there exists a unique minimal closed (in the Zariski topology) subgroup H_0 of H of finite index, which is termed the *connected component* of H. Moreover, in this event H_0 is an irreducible component of H and all other irreducible components of H are cosets of H_0. It seems likely that the theory of algebraic groups over a given group H may well go beyond the point we reached in Section 8. Some possible directions of further development of this theory are indicated at the end of the section and in the open problems. A small collection of open problems forms the final section of the paper.

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2. UNIVERSAL CLASSES

2.1. *Varieties, Quasivarieties, and Prevarieties*

We start with some general remarks on axiomatic description of classes of groups by universal sentences. All the definitions and results of this section make sense (with obvious adjustments) for arbitrary algebraic structures, but for simplicity we focus only on groups.

Let us fix an arbitrary group G. Recall that a *universal sentence* in the language L_G is a formula of the type

$$\forall x_1 \cdots \forall x_n \left(\bigvee_{j=1}^{s} \bigwedge_{i=1}^{t} \left(u_{ji}(\bar{x}, \bar{g}_{ij}) = 1 \wedge w_{ij}(\bar{x}, \bar{f}_{ij}) \neq 1\right)\right),$$

where $\bar{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables, $\bar{g}_{ij}$ and $\bar{f}_{ij}$ are arbitrary tuples of elements (constants) from G, and u_{ij}, w_{ij} are group words.

A class of G-groups $\mathcal{K}$ is *axiomatizable* by a set of universal sentences Σ in the language L_G if $\mathcal{K}$ consists precisely of all G-groups satisfying all formulas from Σ. In this event we say that $\mathcal{K}$ is *a universal class* and Σ is a set of *axioms* for $\mathcal{K}$. For a class of G-groups $\mathcal{K}$ denote by $\mathrm{Th}_{G,\forall}(\mathcal{K})$ the G-universal theory of $\mathcal{K}$, i.e., the set of all universal sentences of L_G which are true in every group from $\mathcal{K}$. If $\mathcal{K} = \{H\}$ then we write $\mathrm{Th}_{G,\forall}(H)$ instead of $\mathrm{Th}_{G,\forall}(\{H\})$ and use this approach in all similar circumstances. Two G-groups H and K are *G-universally equivalent* if $\mathrm{Th}_{G,\forall}(H) = \mathrm{Th}_{G,\forall}(K)$. The *universal closure* of $\mathcal{K}$ is the class $\mathrm{ucl}(\mathcal{K})$ axiomatizable by $\mathrm{Th}_{G,\forall}(\mathcal{K})$. Notice, that $\mathrm{ucl}(\mathcal{K})$ is the minimal universal class containing $\mathcal{K}$. Observe, that the class $\mathrm{ucl}(G)$ consists of all G-groups G-universally equivalent to G

(since G is a subgroup of every group in $\mathrm{ucl}(G)$). This is not the case for an arbitrary class $\mathcal{K}$.

As we have mentioned in the introduction $\mathrm{ucl}(\mathcal{K}) = SP_u(\mathcal{K})$, where $P_u(\mathcal{K})$ is the class of all ultraproducts of groups from $\mathcal{K}$. In particular, $\mathrm{ucl}(H) = SP_u(H)$, and in this case $P_u(H)$ is just the class of all groups isomorphic to ultrapowers of H.

The following lemma shows that under proper circumstances the universal closure of a class of G-groups $\mathcal{K}$ is completely determined by the subclass $\mathcal{K}_\omega$ of all finitely generated G-groups from $\mathcal{K}$. This is a typical result in model theory based on the fact that any algebraic structure is embeddable into an ultraproduct of its finitely generated substructures (we refer to [CK, Mal2] for details). Recall that by $S(\mathcal{K})$ we denote the class of all G-groups G-isomorphic to G-subgroups from $\mathcal{K}$.

LEMMA 1. *Let $\mathcal{K}$ and $\mathcal{M}$ be classes of G-groups. Then the following hold*:

(1) *If $\mathcal{M}$ is a universal class and $S(\mathcal{K})_\omega \subseteq \mathcal{M}$ then $\mathcal{K} \subseteq \mathcal{M}$.*

(2) *If $\mathcal{K}$ and $\mathcal{M}$ are both universal classes then $\mathcal{K} = \mathcal{M}$ if and only if $\mathcal{K}_\omega = \mathcal{M}_\omega$.*

(3) *If $S(\mathcal{K})_\omega \subseteq \mathcal{K}$ then* $\mathrm{ucl}(\mathcal{K}) = \mathrm{ucl}(\mathcal{K}_\omega)$.

Now we consider universal classes which can be axiomatized by universal sentences of a particular type. An identity in the language L_G is a formula of the type

$$\forall x_1 \cdots \forall x_n \left(\bigwedge_{i=1}^{m} r_i(x) = 1 \right), \tag{1}$$

where $r_i(x)$ is a group word in $x_1, \ldots, x_n$ with constants from G. A class of groups $\mathcal{K}$ is called *a variety* if it can be axiomatized by a set of identities. Birkhoff proved [Bir] that $HSP(\mathcal{K})$ is the minimal variety containing $\mathcal{K}$; it is called *the variety generated by* $\mathcal{K}$ (here $H(\mathcal{K})$ and $P(\mathcal{K})$ are the classes of all homomorphic images and all unrestricted Cartesian products of groups from $\mathcal{K}$). Thus, as we mentioned in the Introduction, $\mathrm{var}(\mathcal{K}) = HSP(\mathcal{K})$. By $I_G(\mathcal{K})$ we denote the *equational theory* of a class of groups $\mathcal{K}$, i.e., the set of all identities in the language L_G which are true in every group from $\mathcal{K}$. Naturally, $I_G(\mathcal{K})$ is a set of axioms for $\mathrm{var}(\mathcal{K})$.

We need to consider classes more general then varieties. A *quasi-identity* in the language L_G is a formula of the type

$$\forall x_1 \cdots \forall x_n \left(\bigwedge_{i=1}^{m} r_i(x) = 1 \rightarrow s(x) = 1 \right), \tag{2}$$

where $r_i(x)$ and $s(x)$ are group words in $x_1, \ldots, x_n$ with constants from G. A class of groups $\mathcal{K}$ is called *a quasivariety* if it can be axiomatized by a set of quasi-identities.

For a class of G-groups $\mathcal{K}$ denote by $Q_G(\mathcal{K})$ the set of all quasi-identities in the language L_G which hold in all groups from $\mathcal{K}$. Clearly, $Q_G(\mathcal{K})$ is a set of axioms of the *minimal* quasivariety qvar($\mathcal{K}$) containing $\mathcal{K}$. Again, there are several characterizations of quasivarieties in terms of algebraic operations. Malcev proved [Mal1] that the minimal quasivariety qvar($\mathcal{K}$) generated by a class $\mathcal{K}$ is equal to the class $SP_f(\mathcal{K})$, where $P_f(\mathcal{K})$ is the class of all filter products of algebraic structures from $\mathcal{K}$. Gratzer and Lasker showed [GL] that qvar($\mathcal{K}$) $= SPP_{\mathrm{up}}(\mathcal{K})$ where $P_{\mathrm{up}}(\mathcal{K})$ is the class of all ultrapowers of algebraic structures from $\mathcal{K}$. These results hold for arbitrary algebraic structures.

Observe that every variety is a quasivariety. Indeed, the identity (1) is equivalent to conjunction of quasi identities

$$\forall x_1 \cdots \forall x_n \forall y \big(y = y \rightarrow r_i(x_1, \ldots, x_n) = 1\big), \qquad i = 1, \ldots, m.$$

This also shows that $\mathrm{qvar}(\mathcal{K}) \subseteq \mathrm{var}(\mathcal{K})$.

It follows from the definition that any quasivariety is closed under taking subgroups and Cartesian products; i.e., $SP(\mathcal{K}) \subseteq \mathrm{qvar}(\mathcal{K})$. In general, $SP(\mathcal{K}) \neq \mathrm{qvar}(\mathcal{K})$ (see the discussion of the Malcev's problem in the next section).

A class $\mathcal{K}$ is called *a prevariety* if it is closed under taking subgroups and Cartesian products. It is not hard to see that the class $\mathrm{pvar}(\mathcal{K}) = SP(\mathcal{K})$ is the minimal prevariety containing $\mathcal{K}$; it is called *the prevariety generated by* $\mathcal{K}$. It follows from the discussion above that for any class $\mathcal{K}$

$$\mathrm{pvar}(\mathcal{K}) \subseteq \mathrm{qvar}(\mathcal{K}) \subseteq \mathrm{var}(\mathcal{K}).$$

Remark 1. In general, a prevariety is not an axiomatizable class.

Indeed, for example, let Z be an infinite cyclic group and $\mathcal{K} = \{Z\}$. It is not hard to see that a Cartesian product of Z does not contain a nontrivial divisible subgroup. So there are no nontrivial divisible groups in $SP(\mathcal{K})$. On the other hand, it is known from model theory that there are ultrapowers of Z containing infinite divisible subgroups (see, for example, [CK]). Since every axiomatizable class is closed under taking ultrapowers we deduce that $SP(\mathcal{K})$ is not axiomatizable.

Recall that a class of G-groups $\mathcal{K}$ *G-separates* a G-group H, if for each non-trivial $h \in H$ there exists a G-homomorphism $\phi\colon H \to A$, $A \in \mathcal{K}$, such that $\phi(h) \neq 1$. The following lemma is easy (see [BMR1, discussion in Section 2.4]).

LEMMA 2. *Let $\mathcal{K}$ be a class of G-groups. Then a G-group H belongs to* $\mathrm{pvar}(\mathcal{K})$ *if and only if H is G-separated by $\mathcal{K}$.*

Now we are ready to formulate a first result which ties universal algebra with algebraic geometry over groups.

PROPOSITION A. *Let H be a G-group. Then every group from* $\mathrm{pvar}(H)_\omega$ *is the coordinate group of an algebraic set over H, and conversely, every such coordinate group belongs to* $\mathrm{pvar}(H)_\omega$.

Proof. It follows from the definition that the coordinate group of an algebraic set over H defined by a system with coefficients in G must be finitely generated as a G-group. In the first paper [BMR1] we proved that a finitely generated G-group K is the coordinate group of an algebraic set over H if and only if K is G-separated by H. Now the result follows from Lemma 2.

2.2. *Radicals and the Generalized Nullstellensatz*

Let $\mathcal{Q}$ be an abstract class of G-groups closed under unrestricted Cartesian products and G-subgroups (for example, a variety, a quasivariety, or a prevariety).

A subgroup I of a G-group H is called a *G-ideal* of H, or simply an *ideal*, if I is the kernel of some G-homomorphism $\phi\colon H \to K$. Plainly, a normal subgroup I of H is an ideal if and only if it intersects trivially the distinguished subgroup G of H. An ideal I of H is termed *a radical ideal* with respect to $\mathcal{Q}$ (or a *$\mathcal{Q}$-radical ideal*) if it is the kernel of some G-homomorphism $\phi\colon H \to K$ for which $K \in \mathcal{Q}$. Since $\mathcal{Q}$ is closed under subgroups, we see that an ideal I of H is $\mathcal{Q}$-radical if and only if the quotient group H/I belongs to $\mathcal{Q}$.

For a subset $S \subseteq H$ we denote by $\mathrm{Rad}_{\mathcal{Q}}(S)$ (or, simply, by $\mathrm{Rad}(S)$) the least $\mathcal{Q}$-radical subgroup of H containing S.

LEMMA 3. *Let H be a G-group and $S \subseteq H$. Then*:

(1) *the radical* $\mathrm{Rad}_{\mathcal{Q}}(S)$ *exists and it is equal to the intersection of all $\mathcal{Q}$-radical subgroups of H containing S*;

(2) *any G-homomorphism $\phi\colon H \to K$ gives rise to the canonical G-homomorphism*

$$\phi_{\mathrm{Rad}}\colon H/\mathrm{Rad}_{\mathcal{Q}}(S) \to K/\mathrm{Rad}_{\mathcal{Q}}(\phi(S)).$$

Proof. To prove (1) denote by R the intersection of all $\mathcal{Q}$-radical subgroups of H containing S, say $R = \bigcap_{j\in J} R_j$. Then R is the kernel of the following G-homomorphism:

$$\phi\colon H \to \prod_{j\in J} H/R_j.$$

Since $\mathcal{Q}$ is closed under Cartesian products the ideal R is $\mathcal{Q}$-radical, as desired. (2) is obvious.

Let H and K be G-groups and let S be a subset of H. In [BMR1, Section 4.2] we defined the K-radical $\mathrm{Rad}_K(S)$ of S in H as the intersection of the kernels of all G-homomorphisms $\phi\colon H \to K$ for which $\phi(S) = 1$. Now we give another description of this radical.

LEMMA 4. *Let H and K be G-groups. Then for any subset $S \subseteq H$ the following equality holds*:

$$\mathrm{Rad}_K(S) = R_{\mathrm{pvar}(K)}(S).$$

Proof. Obviously, $R_{\mathrm{pvar}(K)}(S) \subseteq \mathrm{Rad}_K(S)$. Now if $h \in H$ and $h \notin R_{\mathrm{pvar}(K)}(S)$ then there exists a G-homomorphism ϕ: $H \to K_1$, $K_1 \in \mathrm{pvar}(K)$, such that $\phi(h) \neq 1$. Since K_1 is G-separated by K then there exists a G-homomorphism ϕ_1: $K_1 \to K$ such that $\phi_1(\phi(h)) \neq 1$. It follows that $h \notin \ker(\phi \circ \phi_1)$, where $\phi \circ \phi_1$: $H \to K$. Hence $h \notin \mathrm{Rad}_K(S)$, as desired. This proves the lemma.

Remark 2. The notion of a radical ideal plays an important role in classical algebraic geometry. If k is a field and $k[X] = k[x_1, \ldots, x_n]$ is a ring of polynomials over k then for any $S \subseteq k[X]$ the radical $\mathrm{Rad}_k(S)$ is defined as the set of all polynomials that vanish on all roots in k of the system $S(X) = 0$. It is easy to see that $\mathrm{Rad}_k(S)$ is precisely the intersection of the kernels of all homomorphisms ϕ: $k[X] \to k$ for which $\phi(S) = 0$. Hilbert's Nullstellensatz describes the radical $\mathrm{Rad}_k(S)$ in the case when k is algebraically closed.

In view of this remark, by *the generalized Nullstellensatz* with respect to $\mathcal{Q}$ we understand any "effective" description of the radical $R_{\mathcal{Q}}(S)$ in terms of S and $\mathcal{Q}$.

Now we describe radicals in the case when $\mathcal{Q}$ is a variety or a quasivariety. We recall, first, the notion of a verbal subgroup in the language L_G. Let W be a set of group words in variables X and with constants from G. For a G-group H by $W(H)$ denote the (normal) subgroup of H generated by all values of words from W in the group H:

$$W(H) = \langle w(h_1, \ldots, h_n) \mid w \in W, h_i \in H \rangle.$$

$W(H)$ is called the *verbal* subgroup of H with respect to the set of words W. Now suppose $\mathcal{Q}$ is a variety of G-groups. Denote by $W_{\mathcal{Q}}$ the set of group words which occur (as the left-hand sides of equalities) in the identities from $I(\mathcal{Q})$. Thus, if an identity

$$\forall x_1 \cdots \forall x_n \big(r_1(x) = 1 \wedge \cdots \wedge r_m(x) = 1\big)$$

is in $I(\mathcal{Q})$, then the words $r_1, \ldots, r_m$ are in $W_{\mathcal{Q}}$. By definition $R_{\mathcal{Q}}(S)$ is the minimal normal subgroup of H which contains S and all the elements $w(h_1, \ldots, h_n)$, where $w \in W_{\mathcal{Q}}$, $h_i \in H$. It follows that the radical is generated as a normal subgroup by S and $W_{\mathcal{Q}}(H)$:

$$R_{\mathcal{Q}}(S) = \langle S \rangle^H W_{\mathcal{Q}}(H).$$

The subgroup $R_{\mathcal{Q}}(S)$ is called the *$\mathcal{Q}$-verbal* subgroup of H generated by S and its elements are *logical consequences* of S and the set of identities $I(\mathcal{Q})$.

If $\mathcal{Q}$ is a quasivariety then elements of $\mathrm{Rad}_{\mathcal{Q}}(S)$ can be described as logical consequences of S and the set $Q(\mathcal{Q})$ of all the quasi-identities which hold in $\mathcal{Q}$. More precisely, one can describe $R_{\mathcal{Q}}(S)$ in terms of formal derivations in some first order logic (see, for example, [Gor]), but we would like to have a more algebraic definition. To this end, let R_0 be the normal closure of the set S in H. Assume now that a normal subgroup R_i of H is already defined. In this event let R_{i+1} be the minimal normal subgroup of H which contains R_i and also contains every element $h \in H$ for which there are some elements $h_1, \dots, h_n \in H$ and a quasi-identity

$$\forall x_1 \cdots \forall x_n \big((r_1(x) = 1 \wedge \cdots \wedge r_m(x) = 1) \rightarrow s(x) = 1\big)$$

from $Q(\mathcal{Q})$ such that $r_k(h_1, \dots, h_n) \in R_i$ for $k = 1, \dots, m$ and $h = s(h_1, \dots, h_n)$. Thus, by induction, we define a sequence of normal subgroups

$$R_0 \leq R_1 \leq \cdots R_i \leq R_{i+1} \leq \cdots \tag{3}$$

of the group H. Now we can describe the radical of S with respect to $\mathcal{Q}$.

LEMMA 5. *Let H be a G-group and let $\mathcal{Q}$ be a quasivariety of groups. Then for any subset $S \subset H$ the following equality holds*:

$$\mathrm{Rad}_{\mathcal{Q}}(S) = \bigcup_{i=0}^{\infty} R_i.$$

Proof. Clearly, the union $R = \bigcup_{i=0}^{\infty} R_i$ is a normal subgroup of H containing S. To prove $R = \mathrm{Rad}_{\mathcal{Q}}(S)$ it suffices to show that H/R belongs to $\mathcal{Q}$, i.e., all the quasi-identities from $Q(\mathcal{Q})$ hold in H/R. The latter follows from the inductive definition of the sequence (3). This proves the lemma.

The following result is an immediate consequence from the description of the radical in Lemma 5.

PROPOSITION 1. *Let H be a G-group, let $\mathcal{Q}$ be a quasivariety of groups, and let S be a subset of H. Then for any element $r \in \mathrm{Rad}_{\mathcal{Q}}(S)$ there exists a finite subset $S_r \subseteq S$ such that $r \in \mathrm{Rad}_{\mathcal{Q}}(S_r)$.*

Proof. Let $r \in \mathrm{Rad}_{\mathcal{Q}}(S)$. Then $r \in R_i$ for some non-negative integer i. We use induction on i. If $r \in R_0$ then r is in the normal closure of some finite subset of S, as desired. Now suppose that $r \in R_{i+1} - R_i$ for some i. Then there exists a quasi-identity

$$\forall x_1 \cdots \forall x_n \big((r_1(x) = 1 \wedge \cdots \wedge r_m(x) = 1) \rightarrow s(x) = 1\big),$$

in $Q(\mathcal{Q})$ and elements $h_1, \ldots, h_n \in H$ such that $r_k(h_1, \ldots, h_n) \in R_i$ for any $k = 1, \ldots, m$ and $r = s(h_1, \ldots, h_n)$. By induction for every such k there is a finite subset $S_k \subseteq S$ such that $r_k(h_1, \ldots, h_n) \in R(S_k)$. Put $S_r = S_1 \cup \ldots \cup S_n$. It follows that all the elements $r_k(h_1, \ldots, h_n)$ lie in $R(S_r)$, as well as r. This proves the result.

COROLLARY 1. *Let H be a group and let $\mathcal{Q}$ be a quasivariety of groups.*

(1) *Then the union of an ascending chain of $\mathcal{Q}$-radical subgroups of H is a $\mathcal{Q}$-radical subgroup of H.*

(2) *Assume that a subset $S \subseteq H$ is an ascending union of the subsets $S_0 \leq S_1 \leq S_2 \leq \cdots$. Then*

$$R_{\mathcal{Q}}(S) = \bigcup_{i=0}^{\infty} R(S_i).$$

If $\mathcal{Q}$ is a prevariety then, as we mentioned above, $\mathcal{Q}$ is not always axiomatizable. Hence the radicals with respect to $\mathcal{Q}$ cannot be described even in terms of logical consequences. This shows that the generalized Nullstellensatz for prevarieties is extremely difficult.

2.3. *Relatively Free Groups*

Let $\mathcal{Q}$ be either a prevariety, or a quasivariety, or a variety of groups in the language L_G. Assume also that $\mathcal{Q}$ contains a non-trivial group. It is known (see, for example, [Mal2]) that in this event $\mathcal{Q}$ has free objects. We review briefly how to construct $\mathcal{Q}$-free groups.

Let X be a set. Recall that a group $F_{\mathcal{Q}}(X) \in \mathcal{Q}$ is called a *$\mathcal{Q}$-free group* (or a *free group relative to $\mathcal{Q}$*) if $F_{\mathcal{Q}}(X)$ is G-generated by X and for every group $H \in \mathcal{Q}$ any map $X \to H$ extends uniquely to a G-homomorphism $F_{\mathcal{Q}}(X) \to H$. Now, let

$$\mathcal{H} = \{H_i \mid i \in I\}$$

be a set of groups from $\mathcal{Q}$ equipped with maps $\lambda_i\colon X \to H_i$ such that $\lambda_i(X)$ G-generates H_i. Moreover, we may assume that for every group K from $\mathcal{Q}$ and every map $\alpha\colon X \to K$ there exists a group $H_i \in \mathcal{H}$ such that the map $\lambda_i(x) \to \alpha(x)$, $x \in X$, extends to a G-homomorphism of H_i into K. Denote by H_X the unrestricted Cartesian product of groups H_i, $i \in I$:

$$H_X = \prod_{i \in I} H_i.$$

Obviously, $H_X \in \mathcal{Q}$. We identify X with its image in H_X via the diagonal embedding

$$\Delta\colon X \to H_X,$$

where $\Delta(x)(i) = \lambda_i(x)$ for $i \in I$. Finally, let $F_{\mathcal{Q}}(X)$ be the G-subgroup of H_X generated by X. We claim that this group satisfies the universal property of a free group in $\mathcal{Q}$. Indeed, if $H \in \mathcal{Q}$ and μ: $X \to H$ is a map, then there exists $i \in I$ and a G-homomorphism ϕ: $H_i \to H$ which extends the map $\lambda_i(x) \to \mu(x), x \in X$. Denote by π_i: $H_X \to H_i$ the canonical projection of H_X onto H_i. Then the composition of the restriction of π_i onto $F_{\mathcal{Q}}(X)$ and ϕ gives a homomorphism $F_{\mathcal{Q}}(X) \to H$ which extends μ. This shows that $F_{\mathcal{Q}}(X)$ is free in $\mathcal{Q}$ with basis X, as claimed.

PROPOSITION 2. *Let $\mathcal{K}$ be a class of G-groups. Then for any set X the free groups with basis X in the categories* var($\mathcal{K}$), pvar($\mathcal{K}$), *and* qvar($\mathcal{K}$) *are isomorphic*.

Proof. Let us prove, first, that a free group, say $F(X)$, with basis X in pvar($\mathcal{K}$) is also a free group with basis X in var($\mathcal{K}$). To see this let $H \in var(\mathcal{K})$ and $\phi : X \to H$ be a map. Then either $H \in$ pvar($\mathcal{K}$) and in this case ϕ can be extended to a homomorphisms $F(X) \to H$, or H is a homomorphic image of some group $H_1 \in$ pvar($\mathcal{K}$) under a homomorphism ψ: $H_1 \to H$ (this follows from the algebraic description of classes pvar($\mathcal{K}$) and $var(\mathcal{K})$). Since $F(X)$ is free in pvar($\mathcal{K}$) there exists a homomorphism δ: $F(X) \to H_1$ such that the composition $\delta \circ \psi$ is a homomorphism from $F(X)$ into H for which $\delta \circ \psi(x) = \phi(x), x \in X$. This shows that $F(X)$ is a free group in $var(\mathcal{K})$. Since

$$\text{prev}(\mathcal{K}) \subseteq \text{qvar}(\mathcal{K}) \subseteq \text{var}(\mathcal{K})$$

we also have that $F(X)$ is a free group in qvar($\mathcal{K}$). This proves the proposition.

We say that a group $H \in \mathcal{Q}$, generated by X, has *a presentation* $< X \mid S >$ in $\mathcal{Q}$ for some $S \subseteq F_{\mathcal{Q}}(X)$ and write $H = \langle X \mid S \rangle_{\mathcal{Q}}$ if

$$H \simeq F_{\mathcal{Q}}(X) / \operatorname{Rad}_{\mathcal{Q}}(S).$$

If S is finite then $H = \langle X \mid S \rangle_{\mathcal{Q}}$ is *finitely presented* in $\mathcal{Q}$. It follows immediately from the definition of $\operatorname{Rad}_{\mathcal{Q}}(S)$ that for any group $K \in \mathcal{Q}$ and any homomorphism ϕ: $F_{\mathcal{Q}}(X) \to K$ if $S \subseteq \ker(\phi)$ then ϕ induces a unique homomorphism $H \to K$.

Using radicals it is easy to relate a presentation of the group H in the variety of all G-groups with a presentation of H in $\mathcal{Q}$. Indeed, let η: $G[X] \to F_{\mathcal{Q}}(X)$ be the canonical epimorphism from the free G-group $G[X]$ onto the free $\mathcal{Q}$-group $F_{\mathcal{Q}}(X)$. If S is a subset of $G[X]$ and $H = \langle X \mid \eta(S) \rangle_{\mathcal{Q}}$, then $H = \langle X \mid S \rangle$ in the variety of all G-groups. This follows immediately from Lemma 3. As a corollary we have the following result.

LEMMA 6. *Let η: $G[X] \to F_{\mathcal{Q}}(X)$ be the canonical epimorphism. Suppose that $T \subseteq G[X]$ and $S \subseteq F_{\mathcal{Q}}(X)$ are sets such that $S = \eta(T)$. Then the $\mathcal{Q}$-radical of T in $G[X]$ is the complete preimage of the $\mathcal{Q}$-radical of S in $F_{\mathcal{Q}}(X)$.*

3. RADICALS AND MALCEV'S PROBLEMS

One of the principal goals of algebraic geometry over a G-group H is to describe algebraic sets over H, or, equivalently, to describe radicals of subsets of $G[X]$ with respect to pvar(H). Since pvar(H) may not be even axiomatizable the description of these radicals (the generalized Nullstellensatz) might be extremely difficult. The goal of this section is to find conditions on H which, at least, allow one to describe the radicals in terms of logical consequences.

3.1. *q-Compactness and Malcev's Problem*

The first natural attempt to get a logical description of the radicals lies in finding conditions on H which make pvar(H) axiomatizable. The following result is due to Malcev [Mal1].

PROPOSITION 3. *A prevariety is axiomatizable if and only if it is a quasivariety.*

Proof. Obviously, a quasivariety is an axiomatizable prevariety. So one needs only to prove that an axiomatizable prevariety, say $\mathcal{K}$, is a quasivariety. Since $\mathcal{K}$ is a prevariety it is closed under taking subgroups and Cartesian products. Since $\mathcal{K}$ is axiomatizable it is closed under taking ultraproducts. Hence $\mathcal{K} = SPP_u(\mathcal{K})$ which shows that $\mathcal{K}$ is a quasivariety, as needed.

The result above shows that the following question plays an important part here:

(M1) *For which classes $\mathcal{K}$ is the prevariety* pvar$(\mathcal{K})$ *generated by $\mathcal{K}$ a quasivariety?*

This question is known in the theory of quasivarieties as a Malcev's problem [Mal2]. Malcev himself gave the following sufficient condition for the prevariety pvar$(\mathcal{K})$ to be a quasivariety [Mal1].

PROPOSITION 4. *Let $\mathcal{K}$ be an axiomatizable class of groups. Then* pvar$(\mathcal{K})$ *is a quasivariety.*

Sketch of the proof. Due to the algebraic characterization of quasivarieties by Gratzer and Lasker (see Section 1.1) it suffices to show that pvar$(\mathcal{K}) = SP(\mathcal{K})$ is closed under ultrapowers. This comes down to the proof that an ultrapower of a Cartesian product of groups from $\mathcal{K}$ is again a Cartesian product of groups from $\mathcal{K}$ (here one uses that $\mathcal{K}$ is axiomatizable).

Recently this Malcev problem has been solved completely by Gorbunov [Gor]. We need some definitions to explain his solution. Let $S = \{s_1, s_2, \ldots\}$ be an infinite system of equations over H. We allow here an infinite number of different variables involved in S. For example, s_1 may depend on variable x_1, s_2 may contain variables x_1, x_2, and so on. As usual, one can define the solution set of the system $S = 1$ in H. For example, if the set of variables that occur in S is infinite and countable then

$$V_H(S) = \big\{h \in H^{\omega} \mid s(h) = 1 \textit{ for any } s \in S\big\}.$$

An equation $f = 1$ over H is *a consequence* of the system $S = 1$ if $V_H(S) = V_H(S \cup \{f\})$. Following [Gor] we say that a class of groups $\mathcal{K}$ is *q-compact* if for an arbitrary system of equations $S = 1$ if an equation $f = 1$ is a consequence of the system $S = 1$ over every group from $\mathcal{K}$, then there exists a finite subsystem $S_0 = 1$ of $S = 1$ such that $f = 1$ is a consequence of $S_0 = 1$ over every group from $\mathcal{K}$. This notion of q-compactness can be expressed in the form of a compactness theorem for quasi-identities (hence the name). Indeed, recall that a set T of formulas is called *satisfiable* in a class $\mathcal{K}$ if one can assign some elements from a particular group from $\mathcal{K}$ as values to the variables which occur in T in such a way that all formulas from T become true. The set T is called *locally satisfiable* in $\mathcal{K}$ if every finite subset of it is satisfiable in $\mathcal{K}$. It is not hard to see that a class $\mathcal{K}$ is q-compact if and only if every locally satisfiable in $\mathcal{K}$ set of formulas of the type

$$T = \{s = 1 \mid s \in S\} \cup \{f \neq 1\}$$

(here S and f are as above) is satisfiable in $\mathcal{K}$.

It turns out that the prevariety pvar($\mathcal{K}$) is a quasivariety if and only if $\mathcal{K}$ is q-compact [Gor].

The following result shows that q-compactness is a very restrictive condition on a class $\mathcal{K}$.

PROPOSITION 5. *Let H be a group which contains an infinite cyclic subgroup and does not contain any non-trivial divisible Abelian subgroup* (*for example*, *H is an infinite hyperbolic group*). *Then the class $\mathcal{K} = \{H\}$ is not q-compact*.

Proof. Let H be a group containing an infinite cyclic subgroup Z. Then any ultrapower $H^I/\mathcal{D}$ of H, with respect to a non-principal ultrafilter $\mathcal{D}$ on a set I, contains the ultrapower $Z^I/\mathcal{D}$. It is known (see, for example, [CK]) that $Z^I/\mathcal{D}$ contains an infinite divisible subgroup Q. If the class $\{H\}$ is q-compact, then pvar(H) is a quasivariety; hence it is an axiomatizable class, and so it contains the ultrapower $H^I/\mathcal{D}$. Since all groups from $\mathrm{pvar}(H)$ are G-separated by H, it follows that the group $H^I/\mathcal{D}$ together with its subgroup Q are separated by H. This implies that H has a non-trivial divisible subgroup, contradicting the assumptions on H. It follows that the class $\{H\}$ is not q-compact.

3.2. *q_ω-Compactness and the Restricted Malcev Problem*

In the previous section we achieved our goal, but did not get much. Fortunately, if a system of equations $S(X) = 1$ has only finitely many variables the radical $\mathrm{Rad}_H(S)$ depends only on finitely generated G-groups from $\mathrm{pvar}(H)$. This shows that for the purposes of algebraic geometry it is more natural to consider the following variation of the Malcev problem (we call it *the restricted Malcev problem*):

(M2) Describe the classes $\mathcal{K}$ for which $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$.

The lemma below shows that the condition $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$ provides one with a useful description of groups from $\mathrm{qvar}(\mathcal{K})$. Recall that for a class $\mathcal{K}$ by $L(\mathcal{K})$ we denote the class of all G-groups in which all finitely generated G-subgroups belong to $\mathcal{K}$, i.e., *locally* $\mathcal{K}$-groups, and by $\mathrm{Sep}(\mathcal{K})$ we denote the class of G-groups which are G-separated by $\mathcal{K}$.

LEMMA 7. *Let $\mathcal{K}$ be a class of G-groups. Then the following conditions are equivalent*:

(1) $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$;
(2) $\mathrm{qvar}(\mathcal{K}) = LSP(\mathcal{K})$;
(3) $\mathrm{qvar}(\mathcal{K}) = L\,\mathrm{Sep}(\mathcal{K})$.

Proof. Suppose $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$. Then

$$L(\mathrm{pvar}(\mathcal{K})_\omega) = L(\mathrm{qvar}(\mathcal{K})_\omega).$$

Notice that

$$L(\mathrm{pvar}(\mathcal{K})_\omega) = L(SP(\mathcal{K})_\omega) = L(SP(\mathcal{K})) = LSP(\mathcal{K}).$$

On the other hand, $L(\mathrm{qvar}(\mathcal{K})_\omega) = \mathrm{qvar}(\mathcal{K})$. Indeed, if $H \in L(\mathrm{qvar}(\mathcal{K})_\omega)$ then H is a subgroup of some ultraproduct of groups from $\mathrm{qvar}(\mathcal{K})_\omega$. Hence H is in $\mathrm{qvar}(\mathcal{K})$. If $H \in \mathrm{qvar}(\mathcal{K})$ then every finitely generated G-subgroup of H is in $\mathrm{qvar}(\mathcal{K})_\omega$. This implies that $H \in L(\mathrm{qvar}(\mathcal{K})_\omega)$, as claimed. It follows that $LSP(\mathcal{K}) = \mathrm{qvar}(\mathcal{K})$.

Assume now that $LSP(\mathcal{K}) = \mathrm{qvar}(\mathcal{K})$. Hence $LSP(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$. Observe that $LSP(\mathcal{K})_\omega = SP(\mathcal{K})_\omega$. Therefore $SP(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$, as desired. Thus, (1) is equivalent to (2). Observe that (3) is equivalent to (2) since by Lemma 2 the class $SP(\mathcal{K})$ consists precisely of groups G-separated by $\mathcal{K}$. This proves the lemma.

Now one can rephrase the restricted Malcev problem in the following, more traditional way.

(M2*) Describe classes $\mathcal{K}$ for which $\mathrm{qvar}(\mathcal{K}) = LSP(\mathcal{K})$.

The restricted Malcev's problem has a solution similar to the original one. In this case we need the following variation of q-compactness (in which we allow only a given finite set of variables in all the equations from S).

DEFINITION 1. A class $\mathcal{K}$ is called *q_ω-compact* if for each positive integer n and an arbitrary system of equations $S(x_1, \dots, x_n) = 1$ if an equation $r = 1$ is a consequence of the system $S = 1$ over every group from $\mathcal{K}$, then there exists a finite subsystem $S_0 = 1$ of $S = 1$ such that $r = 1$ is a consequence of $S_0 = 1$ over every group from $\mathcal{K}$.

One can formulate the q_ω-compactness in terms of infinite formulas. If an infinite formula

$$\forall X\left(\bigwedge_{s\in S} s(X) = 1 \to r(X) = 1\right)$$

holds in every group from $\mathcal{K}$ then for some finite subsystem $S_0 = 1$ of $S = 1$ the quasi-identity

$$\forall X\left(\bigwedge_{s\in S_0} s(X) = 1 \to r(X) = 1\right)$$

also holds in every group from $\mathcal{K}$.

THEOREM B1. *Let G be a group and let $\mathcal{K}$ be a class of G-groups. Then* $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$ *if and only if $\mathcal{K}$ is q_ω-compact.*

Proof. Let $\mathcal{K}$ be a q_ω-compact class. Since $\mathrm{pvar}(\mathcal{K})_\omega \subseteq \mathrm{qvar}(\mathcal{K})_\omega$ one needs only to prove the reverse inclusion. To this end assume that $H \in \mathrm{qvar}(\mathcal{K})_\omega$. Consider a finitely generated presentation of H in $\mathrm{qvar}(\mathcal{K})$,

$$H = \langle X \mid S(X)\rangle,$$

where $X = \{x_1, \dots, x_n\}$ is a finite set and $S(X)$ is a set of words in X with constants from G. Let H_p be a group with the same presentation but in the prevariety $\mathrm{pvar}(\mathcal{K})_\omega$. Observe that

$$H \simeq F(X)/R_q(S), \qquad H_p \simeq F(X)/R_p(X),$$

where $F(X)$ is a free G-group with basis X in $\mathrm{qvar}(\mathcal{K})$, as well as in $\mathrm{pvar}(\mathcal{K})$ (by Proposition 2), and $R_q(S)$, $R_p(S)$ are radicals of S in the corresponding classes. We claim that for any element $r \in R_p(S)$ the infinite formula

$$q_{S,r} = \forall X\left(\bigwedge_{s\in S} s(X) = 1 \to r(X) = 1\right)$$

holds in any group from $\mathrm{pvar}(\mathcal{K})$. Indeed, let U be an arbitrary group from $\mathrm{pvar}(\mathcal{K})$ and let $u_1, \dots, u_n$ be elements from U which satisfy the conjunction $\bigwedge_{s \in S} s(X) = 1$. The map

$$\phi_0 \colon x_1 \to u_1, \dots, x_n \to u_n$$

extends to a G-homomorphism $\phi \colon F(X) \to U$ (the universal property of the free group $F(X)$). Obviously, $S \subseteq \ker(\phi)$ and hence $R_p(S) \subseteq \ker(\phi)$. It follows that ϕ_0 gives rise to a G-homomorphism $H_p \to U$, and therefore $r(u_1, \dots, u_n) = 1$ in U. This proves the claim. In particular, we see that the statement $q_{S,r}$ holds in every group from $\mathcal{K}$. Since $\mathcal{K}$ is q_ω-compact there exists a finite subset $S_0 \subseteq S$ such that the quasi-identity

$$q_{S_0,r} = \forall X \Big(\bigwedge_{s \in S_0} s(X) = 1 \to r(X) = 1 \Big)$$

holds in $\mathcal{K}$, and therefore it holds in $\mathrm{qvar}(\mathcal{K})$. It follows that r is a logical consequence of S_0, as well as S, in $\mathrm{qvar}(\mathcal{K})$, so $r \in R_q(S)$. This implies that $R_q(S) = R_p(S)$, and consequently, $H = H_p$. Therefore, $H \in \mathrm{pvar}(G)$, as needed.

Suppose now that $\mathrm{pvar}(\mathcal{K})_\omega = \mathrm{qvar}(\mathcal{K})_\omega$. Let $S(X) = 1$ be an arbitrary system of equations in variables $X = \{x_1, \dots, x_n\}$ and with constants from G. Suppose also that for some equation $r(X) = 1$ the infinite formula $q_{S,r}$ holds in every group from $\mathcal{K}$. Then, obviously, $q_{S,r}$ holds in every subgroup of a group from $\mathcal{K}$ as well as in every Cartesian product of groups from $\mathcal{K}$. It follows that $q_{S,r}$ holds in $\mathrm{pvar}(\mathcal{K})$. Hence it holds in $\mathrm{qvar}(\mathcal{K})_\omega$. Since the set of variables X is finite the group $F(X)$ is finitely generated and therefore the radical $R_q(S)$ is completely defined by the class $\mathrm{qvar}(\mathcal{K})_\omega$. This shows that $r \in R_q(S)$. By Proposition 1 there exists a finite subset $S_0 \subseteq S$ such that $r \in R_q(S_0)$. It implies that r is a logical consequence of S_0 in all groups from $\mathcal{K}$, as needed. This proves the theorem.

COROLLARY 2. *Let $\mathcal{K}$ be a q_ω-compact class. Then for any finitely generated G-group H and any subset $S \subseteq H$ the radical of S with respect to* $\mathrm{pvar}(\mathcal{K})$ *is equal to the radical of S with respect to* $\mathrm{qvar}(\mathcal{K})$.

It turns out that many classes of groups are not q-compact, but they are q_ω-compact. Equationally Noetherian groups provide many examples of this kind. Recall that a G-group H is called *G-equationally Noetherian* if every system of equations with coefficients in G and finitely many variables is equivalent over H to some finite part of it. Plainly, every G-equationally Noetherian group is q_ω-compact; i.e., the class $\{H\}$ is q_ω-compact.

COROLLARY 3. *The generalized Nullstellensatz holds for every G-equationally Noetherian group.*

As we mentioned in the Introduction, every linear group over a unitary Noetherian commutative ring is G-equationally Noetherian. In particular, every finitely generated nilpotent group is G-equationally Noetherian (for an arbitrary choice of the distinguished subgroup G!).

Now we give an example of a 2-nilpotent group which is not q_ω-compact (of course, this group is not finitely generated).

EXAMPLE 1. Let G be a nilpotent group of class 2 given by the following presentation (in the variety of class ≤ 2 nilpotent groups):

$$G = \langle a_i, b_i (i \in N) \mid [a_i, a_j] = 1, [b_i, b_j] = 1, [a_i, b_j] = 1 (i \neq j) \rangle.$$

Then the infinite quasi identity

$$\forall x \forall y \left(\bigwedge_{i \in N} \left([x, a_i] = 1 \bigwedge_{j \in N} [x, b_j] = 1 \right) \to [x, y] = 1 \right)$$

holds in G, but for any finite subsets I, J of N the following quasi-identity does not hold in G:

$$\forall x \forall y \left(\bigwedge_{i \in I} \left([x, a_i] = 1 \bigwedge_{j \in J} [x, b_j] = 1 \right) \to [x, y] = 1 \right).$$

Indeed, for any $m \notin I \cup J$ the element $x = a_m$ commutes with all $a_i, i \in I$ and all $b_j, j \in J$, but it is not central in G. Now we give an example from [BMRom] of a finitely generated not q_ω-compact group.

EXAMPLE 2. Let $W = U \wr T$ be the wreath product of a non-Abelian group U by an infinite group T. Denote by B the normal closure of U in W. Then the infinite quasi-identity

$$\forall x \left(\bigwedge_{a, b \in B} [a, b^x] = 1 \to x = 1 \right) \tag{4}$$

holds in W, but any quasi-identity of the type

$$\forall x \left(\bigwedge_{a \in A_0,\, b \in B_0} [a, b^x] = 1 \to x = 1 \right), \tag{5}$$

where A_0, B_0 are finite subsets of B, does not hold in W.

To see this, notice that there is no $x \in W$ which satisfies all the equations

$$[a, b^x] = 1, \qquad a, b \in B.$$

Hence the infinite quasi-identity (4) holds in W. But for any finite system $[a, b^x] = 1,\ a \in A_0, b \in B_0$, there exists a non-trivial $x \in W$ which satisfies this system; i.e., none of the quasi-identities (5) hold in W.

One can find more examples of this type in [BMRom].

At the end of this section we remark that the argument given in the proof of Theorem B1 shows also that, in any event, the classes $\mathrm{pvar}(\mathcal{K})$ and $\mathrm{qvar}(\mathcal{K})$ have precisely the same finitely presented groups.

3.3. *u-Compactness and Universal Classes*

In this section we define a notion of u-compactness of a class of groups in such a way that u-compact classes have a very simple description of their universal closures.

DEFINITION 2. A class $\mathcal{K}$ is called u-compact if for each positive integer n, an arbitrary system of equations $S(X) = 1$ in variables from $X = \{x_1, \ldots, x_n\}$ and constants from G, and finitely many equations $f_1(X) = 1, \ldots, f_m(X) = 1$, if an (infinite) formula

$$\forall X\left(\bigwedge_{s\in S} s(X) = 1 \rightarrow \bigvee_{i=1}^{m} f_i(X) = 1\right)$$

holds in every group from $\mathcal{K}$, then for some finite subsystem $S_0 = 1$ of $S = 1$ the formula

$$\forall X\left(\bigwedge_{s\in S_0} s(X) = 1 \rightarrow \bigvee_{i=1}^{m} f_i(X) = 1\right)$$

also holds in every group from $\mathcal{K}$.

Again, one can formulate u-compactness of $\mathcal{K}$ in terms of the satisfiability of some particular locally satisfiable sets of formulas. Namely, $\mathcal{K}$ is u-compact if and only if a set of the type

$$\{s(X) = 1 \mid s \in S\} \cup \{f_1(X) \neq 1, \ldots, f_m(X) \neq 1\}$$

is locally satisfiable in $\mathcal{K}$ if and only if it is satisfiable in $\mathcal{K}$.

Observe that every G-equationally Noetherian group H is u-compact (i.e., the singleton $\{H\}$ is u-compact). Also, every u-compact class is q_ω-compact, so there are classes which are not u-compact.

As we saw in the previous section (Lemma 7) a class $\mathcal{K}$ is q_ω-compact if and only if the quasivariety generated by $\mathcal{K}$ consists precisely of the G-groups locally G-separated by $\mathcal{K}$. A similar result holds for the universal closure $\operatorname{ucl}(\mathcal{K})$. Recall that a G-group H is G-discriminated by $\mathcal{K}$ if for each finite subset $\{h_1, \ldots, h_n\}$ of non-trivial elements of H there exists a G-homomorphism $\phi: H \to A$, $A \in \mathcal{K}$, such that $\phi(h_j) \neq 1$, $j = 1, \ldots, n$. A detailed discussion of this notion is contained in [BMR1, BMR2]. Denote by $\operatorname{Dis}(\mathcal{K})$ the class of all groups G-discriminated by $\mathcal{K}$. Then $L\operatorname{Dis}(\mathcal{K})$ is the class of groups locally G-discriminated by $\mathcal{K}$.

THEOREM B2. *Let $\mathcal{K}$ be a class of G-groups. Then $\mathcal{K}$ is u-compact if and only if* $\operatorname{ucl}(\mathcal{K}) = L\operatorname{Dis}(\mathcal{K})$.

Proof. Suppose a class of G-groups $\mathcal{K}$ is u-compact. Let $H \in \mathrm{ucl}(\mathcal{K})$ and H_0 be a finitely generated G-subgroup of H. Assume that $H_0 = \langle X \mid S(X) \rangle$ is a presentation of H_0 with respect to a finite set of generators X. If H_0 is not G-discriminated by $\mathcal{K}$ then there are non-trivial elements $f_1(X), \ldots, f_m(X) \in H_0$ such that for any homomorphism ϕ: $H_0 \to A$ with $A \in \mathcal{K}$ at least one of the images $\phi(f_i)$ is trivial. It follows then that the (infinite) formula

$$\forall X \left(\bigwedge_{s \in S} s(X) = 1 \to \bigvee_{i=1}^{m} f_i(X) = 1 \right)$$

holds in every group from $\mathcal{K}$. Since $\mathcal{K}$ is u-compact there exists a finite subsystem $S_0 = 1$ of $S = 1$ for which the universal formula

$$\Psi = \forall X \left(\bigwedge_{s \in S_0} s(X) = 1 \to \bigvee_{i=1}^{m} f_i(X) = 1 \right)$$

also holds in every group from $\mathcal{K}$. Therefore, Ψ holds in every group from the universal closure of $\mathcal{K}$, in particular, in H_0—a contradiction. This shows that H_0 is G-discriminated by $\mathcal{K}$ and, consequently, $\mathrm{ucl}(\mathcal{K}) = L\,\mathrm{Dis}(\mathcal{K})$.

Suppose now that $\mathrm{ucl}(\mathcal{K}) = L\,\mathrm{Dis}(\mathcal{K})$, but the class $\mathcal{K}$ is not u-compact. Then there exists a finite set $X = \{x_1, \ldots, x_n\}$ and a set of formulas

$$T = \left\{ s(X) = 1 \mid s \in S \right\} \cup \left\{ f_1(X) \neq 1, \ldots, f_m(X) \neq 1 \right\}$$

which is locally satisfiable in $\mathcal{K}$, but which is not satisfiable in $\mathcal{K}$. By the classical compactness theorem this set is satisfiable in an ultraproduct U of groups from $\mathcal{K}$. The group U belongs to $\mathrm{ucl}(\mathcal{K})$. Since X is finite the set T is satisfiable in some finitely generated G-subgroup $U_0 \leq U$, which is also in $\mathrm{ucl}(\mathcal{K})$, on a tuple of elements, say $a = (a_1, \ldots, a_n)$. Since $U_0 \in \mathrm{Dis}(\mathcal{K})$ there exists a G-homomorphism ϕ: $U_0 \to A$, $A \in \mathcal{K}$, such that at least one of the images $\phi(f_i(a))$ is non-trivial. Obviously, the elements $(\phi(a_1), \ldots, \phi(a_n))$ satisfy the the set T in A—a contradiction. This proves the theorem.

Notice that two G-group H and K are G-universally equivalent if and only if $\mathrm{ucl}(H) = \mathrm{ucl}(K)$. Now, the theorem above implies the following generalization of the criterion of G-universal equivalence from [BMR1].

COROLLARY 4. *Let H and K be u-compact G-groups. Then H is G-universally equivalent to K if and only if H is locally G-discriminated by K and K is locally G-discriminated by H.*

Since every equationally Noetherian group is u-compact, and a group G-universally equivalent to a G-equationally Noetherian group is again G-equationally Noetherian, we have the following result from [BMR1].

COROLLARY 5. *Let H and K be G-groups and suppose that at least one of them is G-equationally Noetherian. Then H is G-universally equivalent to K if and only if H is locally G-discriminated by K and K is locally G-discriminated by H.*

4. GEOMETRICAL EQUIVALENCE

4.1. *Definitions and Basic Properties*

Fix an arbitrary group G. Let X be a finite set and let $G[X]$ be a free G-group with the basis X.

DEFINITION 3. G-groups H and K are geometrically equivalent if for any finite set X and any subset $S(X) \subseteq G[X]$ the equality $\mathrm{Rad}_H(S) = \mathrm{Rad}_K(S)$ holds.

In the case $G = 1$ one has Plotkin's definition from [Plot1]. The following two propositions (which are due to Plotkin [Plot1]) establish some basic properties of geometrically equivalent groups.

PROPOSITION 6. *If G-groups H and K are geometrically equivalent then their categories of algebraic sets defined by equations with coefficients from G are equivalent.*

Proof. If H and K are geometrically equivalent then for any $S \subseteq G[X]$ the algebraic sets $V_H(S)$ and $V_K(S)$ have exactly the same coordinate groups,

$$G[X]/\,\mathrm{Rad}_H(S) = G[X]/\,\mathrm{Rad}_K(S);$$

i.e., the categories of coordinate groups (defined by equations with coefficients from G) over H and over K coincide. This implies that the categories of algebraic sets over H and over K (defined by equations with coefficients from G) are equivalent (see [BMR1, Plot1] for details), as desired.

PROPOSITION 7. *Let G be a group. Then G-groups H and K are geometrically equivalent if and only if* $\mathrm{pvar}(H)_\omega = \mathrm{pvar}(K)_\omega$.

Proof. Let G-groups H and K be geometrically equivalent. If H_1 is a finitely generated G-group from $\mathrm{pvar}(H)$ then H_1 has a presentation $H_1 = \langle X \mid S\rangle$ with a finite set X. Since H and K are geometrically equivalent we have $\mathrm{Rad}_H(S) = \mathrm{Rad}_K(S)$. Therefore

$$H_1 = G[X]/R_H(S) = G[X]/R_K(S),$$

and H_1 lies in the prevariety $\mathrm{pvar}(K)$.

Suppose now that $\mathrm{pvar}(H)_\omega = \mathrm{pvar}(K)_\omega$. Let X be a finite set and $S \subseteq G[X]$. Then $\Gamma_H(S) = G[X]/\mathrm{Rad}_H(S)$ belongs to $\mathrm{pvar}(H)_\omega$, and hence, to $\mathrm{pvar}(K)_\omega$. It follows that $\Gamma_H(S)$ is G-separated by K. Therefore $\mathrm{Rad}_H(S) = \mathrm{Rad}_K(S)$. This shows that H and K are G-geometrically equivalent, as needed.

COROLLARY 6. *If groups H and K are geometrically equivalent then* $\mathrm{qvar}(H) = \mathrm{qvar}(K)$.

Proof. If H and K are geometrically equivalent then by Proposition 7 $\mathrm{pvar}(H)_\omega = \mathrm{pvar}(K)_\omega$. Obviously, $\mathrm{qvar}(H) = \mathrm{qvar}(\mathrm{pvar}(H))$. By Lemma 1 $\mathrm{qvar}(\mathrm{pvar}(H)) = \mathrm{qvar}(\mathrm{pvar}(H)_\omega)$. It follows that

$$\mathrm{qvar}(H) = \mathrm{qvar}(\mathrm{pvar}(H)_\omega) = \mathrm{qvar}(\mathrm{pvar}(K)_\omega) = \mathrm{qvar}(K),$$

as claimed.

4.2. *Plotkin's Problems*

In this section we discuss [Plot1, Problems 6 and 7]. We formulate them as follows.

(P1) Let H and K be G-groups and let $\mathrm{qvar}(H) = \mathrm{qvar}(K)$. Does this imply that H and K are geometrically equivalent?

Recall that two G-groups are *elementarily equivalent* if they satisfy exactly the same sentences in the language L_G.

(P2) Let H and K be two elementarily equivalent G-groups. Does this imply that H and K are geometrically equivalent?

The theorem below gives a negative answer to both problems above.

THEOREM C1. *Let H be a G-group which is not q_ω-compact. Then there exists an ultrapower K of H such that H and K are not geometrically equivalent.*

Proof. Let H be as above. By Theorem B1 $\mathrm{pvar}(H)_\omega \neq \mathrm{qvar}(H)_\omega$. This implies (see Section 1.1) that

$$SP(H)_\omega \neq SPP_{\mathrm{up}}(H)_\omega.$$

Therefore, there exists an ultrapower K of H such that

$$SP(H)_\omega \neq SP(K)_\omega,$$

which is equivalent to $\mathrm{pvar}(H)_\omega \neq \mathrm{pvar}(K)_\omega$. By Proposition 7 the groups H and K are not geometrically equivalent. This proves the theorem.

COROLLARY 7. *Groups H and K from Theorem C1 show that if H is not q_ω-compact then always there exists a group K for which Plotkin's problems above have a negative answer.*

Indeed, every ultrapower K of H is elementarily equivalent to H; in particular, $\text{qvar}(H) = \text{qvar}(K)$. This gives the result.

One can find even finitely generated groups which provide a negative answer to the problems above. Recall that there are finitely generated not q_ω-compact groups (see Example 2 in Section 2.2).

COROLLARY 8. *If H is a finitely generated group which is not q_ω-compact, then there exists a finitely generated group K_1 such that* $\text{qvar}(H) = \text{qvar}(K_1)$ *but H and K_1 are not geometrically equivalent.*

Proof. To see this, let H be a finitely generated G-group which is not q_ω-compact. Let K be the ultrapower of H from Theorem C1. Observe that H is canonically embedded into K, so we can view K as an H-group. Since $\text{pvar}(H)_\omega \neq \text{pvar}(K)_\omega$ there exists a Cartesian product C of the group K and a finitely generated G-subgroup K_0 of C such that $K_0 \notin \text{pvar}(H)$. Again, the product C can be viewed as an H-group via the diagonal embedding of K into C. Finally, let K_1 be the subgroup of C generated by H and K_0. Then K_1 is a finitely generated G-group and $\text{pvar}(H) \neq \text{pvar}(K_1)$, so H and K_1 are not geometrically equivalent. But $H \in \text{qvar}(K_1)$ as well as $K_1 \in \text{qvar}(H)$. Hence $\text{qvar}(H) = \text{qvar}(K_1)$, as desired.

Now we turn to the question of what conditions the equality $\text{qvar}(H) = \text{qvar}(K)$ imposes onto groups H and K in terms of radicals.

DEFINITION 4. G-groups H and K are called ω-geometrically equivalent if for any finite set X and any finite subset $S \subseteq G[X]$ the equality $R_H(S) = R_K(S)$ holds.

Obviously, geometrically equivalent groups are also ω-geometrically equivalent.

THEOREM C2. *Let H and K be G-groups. Then* $\text{qvar}(H) = \text{qvar}(K)$ *if and only if H and K are ω-geometrically equivalent.*

Proof. Suppose that $\text{qvar}(H) = \text{quar}(K)$. Let X be a finite set and let S be a finite subset of $G[X]$. Then for every $f \in G[X]$ the quasi-identity

$$q_{S,f} = \forall X \left(\bigwedge_{s \in S} s(x) = 1 \to f(x) = 1 \right)$$

holds in H if and only if it holds in K. Hence $\text{Rad}_H(S) = \text{Rad}_K(S)$. This shows that H and K are ω-geometrically equivalent.

Let H and K be ω-geometrically equivalent groups. Assume that $\mathrm{qvar}(H) \neq \mathrm{qvar}(K)$. Then there exists a quasi-identity

$$q_{S,f} = \forall X\Bigg(\bigwedge_{s\in S} s(x) = 1 \to f(x) = 1\Bigg),$$

as above, which holds in H and does not hold in K. This implies that $f \in \mathrm{Rad}_H(S)$, but $f \notin \mathrm{Rad}_K(S)$. Hence $\mathrm{Rad}_H(S) \neq \mathrm{Rad}_K(S)$—contradicting the condition above. This proves the theorem.

4.3. *Plotkin's Problems for q_ω-Compact Groups*

In the previous section we saw that the two Plotkin problems have negative solutions outside the class of q_ω-compact groups. In this section we show that these problems have positive answers inside the class of q_ω-compact groups. As before, we fix an arbitrary group G.

THEOREM C3. *Let H and K be q_ω-compact G-groups. Then H and K are geometrically equivalent if and only if* $\mathrm{qvar}(H) = \mathrm{qvar}(K)$.

Proof. By Proposition 7 H and K are geometrically equivalent if and only if $\mathrm{pvar}(H)_\omega = \mathrm{pvar}(K)_\omega$. By Theorem B1 for q_ω-compact G-groups the equality above is equivalent to the equality $\mathrm{qvar}(H)_\omega = \mathrm{qvar}(K)_\omega$. By Lemma 1 the latter is equivalent to the equality $\mathrm{qvar}(H) = \mathrm{qvar}(K)$, as desired.

As we have mentioned above every equationally Noetherian group is q_ω-compact. Since Abelian groups and linear groups are equationally Noetherian we deduce the following corollaries of Theorem C3.

COROLLARY 9. *Let H and K be Abelian or linear groups. Then H and K are geometrically equivalent if and only if* $\mathrm{qvar}(H) = \mathrm{qvar}(K)$.

By Theorems C1 and C3 there is no way to describe geometrically equivalent groups in terms of axioms of first order logic, unless they are q_ω-compact. This makes it very interesting to try to figure out which groups are q_ω-compact. Since Abelian groups are q_ω-compact then nilpotent groups form the next natural class to study.

Now we prove two results which are interesting only in the case $G \neq 1$. We begin with a simple lemma. Recall that we work with the language L_G, i.e., all the groups we consider are G-groups.

LEMMA 8. *Let G be a q_ω-compact group. Then every group from* $\mathrm{qvar}(G)$ *is q_ω-compact.*

Proof. Let $H \in \mathrm{qvar}(G)$. Since $G \leq H$ we have $\mathrm{qvar}(H) = \mathrm{qvar}(G)$ and $\mathrm{pvar}(G) \subseteq \mathrm{pvar}(H)$. Now, taking subclasses of finitely generated G-groups in the classes above we see that

$$SP(G)_\omega \subseteq SP(H)_\omega \subseteq \mathrm{qvar}(H)_\omega = \mathrm{qvar}(G)_\omega = SP(G)_\omega$$

(the last equality follows from q_ω-compactness of G). So all the classes above are equal, in particular, $SP(H)_\omega = \text{qvar}(H)_\omega$, which implies that H is q_ω-compact (Theorem B1). This proves the lemma.

THEOREM C4. *Let G be a q_ω-compact group. Then any two groups from* qvar(G) *are geometrically equivalent. And vice versa, if any two groups from* qvar(G) *are geometrically equivalent then G is q_ω-compact.*

Proof. If G is q_ω-compact then any two groups from qvar(G) are also q_ω-compact (Lemma 8) and the quasivarieties generated by them are both equal to qvar(G). Hence by Theorem C3 the groups are geometrically equivalent. This proves the first part of the theorem. The reverse is also true. Indeed, suppose G is not q_ω-compact. Then there exists a finitely generated G-group K in qvar(G) which is not in pvar(G). Since K and G are geometrically equivalent we have by Theorem B1 that $\text{pvar}(K)_\omega = \text{pvar}_\omega(G)$. Obviously, $K \in \text{pvar}(K)_\omega$, so $K \in \text{pvar}_\omega(G)$—a contradiction. This shows that G is q_ω-compact.

5. EQUATIONALLY NOETHERIAN GROUPS

Let fix an arbitrary group G. As we have mentioned already a G-group H is called *G-equationally Noetherian* if every system of equations with coefficients in G and finitely many variables is equivalent over H to some finite part of the system. Equivalently, one can define a G-equationally Noetherian group as a group over which the Zariski topology is Noetherian [BMR1]. In this section we give a characterization of G-equationally Noetherian groups in terms of quasivarieties.

THEOREM D1. *Let H be a G-group. Then the following conditions are equivalent:*

(1) *H is G-equationally Noetherian;*

(2) *every finitely generated G-group in the quasivariety* qvar(H) *is finitely presented in* qvar(H);

(3) *every free G-group $F_H(X)$ of finite rank in* qvar(H) *satisfies the ascending chain condition on* qvar(H)*-radical subgroups*;

(4) *every free G-group $G[X]$ of finite rank satisfies the ascending chain condition on* qvar(H)*-radical subgroups.*

Moreover, if H is G-equationally Noetherian then every G-group in qvar(H) *is G-equationally Noetherian.*

Proof. Let H be a G-equationally Noetherian G-group. We prove first that every G-group $K \in \text{qvar}(H)$ is G-equationally Noetherian. Let $S(X) =$

1 be an arbitrary system of equations with coefficients from G on a finite set of variables $X = (x_1, \dots, x_n)$. Since H is G-equationally Noetherian, we have $V_H(S) = V_H(S_0)$ for some finite subset $S_0 = \{s_1, \dots, s_m\}$ of S. Therefore, the following quasi-identity holds in H for any $s(X) \in S(X)$:

$$q_s = \forall X \left(\bigwedge_{i=1}^{m} s_i(X) = 1 \to s(X) = 1 \right).$$

Hence q_s holds in each $K \in \text{qvar}(H)$ for any $s(X) \in S(X)$, so $V_K(S) = V_K(S_0)$. This shows that K is G-equationally Noetherian. Observe from the proof that $S(X) = 1$ is equivalent to $S_0(X) = 1$ over any group K in qvar(H).

Now let K be any group from qvar(H) finitely generated as a G-group and suppose K has the following presentation in qvar(H)

$$K = \langle X \mid S(X) \rangle \simeq F_H(X)/\operatorname{Rad}_H(S),$$

where $F_H(X)$ is a free group in qvar(H) with basis X and $\operatorname{Rad}_H(S)$ is the radical of S with respect to qvar(H). As we mentioned in Section 2.2

$$K = \langle X \mid T \rangle \simeq G[X]/\operatorname{Rad}_H(T),$$

where T is the complete preimage in $G[X]$ of the set S with respect to the canonical epimorphism $\eta\colon G[X] \to F_H(X)$. We claim that if T is equivalent over H to a finite subset $T_0 = \{t_1, \dots, t_m\} \subseteq T$ then $\operatorname{Rad}_H(T) = \operatorname{Rad}_H(T_0)$. Indeed, consider an arbitrary G-homomorphism $\phi\colon G[X] \to K$ with $T_0 \subseteq \ker(\phi)$. Then $t_1(X^\phi) = 1, \dots, t_m(X^\phi) = 1$, so X^ϕ is a solution of the system $T_0(X) = 1$ in K. Since $T(X) = 1$ is equivalent to $T_0(X) = 1$ over K we see that $T(X) \subseteq \ker(\phi)$. This shows that $T \subseteq \operatorname{Rad}_H(T_0)$; hence $\operatorname{Rad}_H(T) = \operatorname{Rad}_H(T_0)$. By Lemma 6

$$\operatorname{Rad}_H(S) = \eta(\operatorname{Rad}_H(T)) = \eta(\operatorname{Rad}_H(T_0)) = \operatorname{Rad}_H(\eta(T_0)),$$

so the group

$$K = F_H(X)/\operatorname{Rad}_H(S) = F_H(X)/\operatorname{Rad}_H(\eta(T_0))$$

is finitely presented in qvar(H).

Suppose now that every finitely generated G-group in qvar(H) is finitely presented in qvar(H). Let $T(X) = 1$ be any system of equations in variables $X = \{x_1, \dots, x_n\}$ with coefficients in G and let $S(X) = \eta(T) \subseteq F_H(X)$. The group $K = F_H(X)/\operatorname{Rad}_H(S)$ belongs to qvar(H). Hence it is finitely presented in qvar(H); i.e., $\operatorname{Rad}_H(S) = \operatorname{Rad}_H(U)$ for some finite set $U \subseteq \operatorname{Rad}_H(S)$. As we have mentioned in Section 2.2, in this event there exists a finite subset $S_0 \subseteq S$ such that $U \subseteq R(S_0)$ and consequently, $\operatorname{Rad}_H(S) = \operatorname{Rad}_H(S_0)$. Denote by T_0 a finite subset of T such that $\eta(T_0) = S_0$. By

Lemma 6 $\mathrm{Rad}_H(T) = \mathrm{Rad}_H(T_0)$. This shows that $V_H(T) = V_H(T_0)$. Indeed, if $V_H(T_0) = \emptyset$ then $V_H(T) = \emptyset = V_H(T_0)$, as desired. Now, suppose $V_H(T_0) \neq \emptyset$ and let $v = (v_1, \ldots, v_n) \in V_H(T_0)$. Then the map $x_i \to v_i$ ($i = 1, \ldots, n$) extends to a G-homomorphism $G[X]/\mathrm{Rad}_H(T_0) \to H$. Therefore all elements $t \in T$ vanish at v; hence $v \in V_H(T)$, and $V_H(S) = V_H(S_0)$. It follows that H is G-equationally Noetherian.

Thus we proved that (1) is equivalent to (2). Equivalence of (2) and (3) is obvious, and equivalence of (3) and (4) follows from Lemma 6. This proves the theorem.

6. COORDINATE GROUPS

6.1. *Coordinate Groups of Affine Spaces*

In this section for a G-group H we describe the coordinate group of the affine space H^n, where n is a positive integer.

PROPOSITION 8. *Let H be a G-group. Then for any positive integer n the coordinate group of the affine space H^n is G-isomorphic to the free group of rank n in the variety $var(H)$.*

Proof. The affine space H^n is an algebraic set defined, for example, by the trivial equation $1 = 1$. Let $F_H(X)$ be a free group in the prevariety $\mathrm{pvar}(H)$ with the basis $X = \{x_1, \ldots, x_n\}$. Then

$$F_H(X) \simeq G[X]/\mathrm{Rad}(T(X))$$

for some $T(X) \subseteq G[X]$. Since $F_H(X)$ is free in $\mathrm{pvar}(H)$ then for any $h = (h_1, \ldots, h_n) \in H^n$ the map $x_1 \to h_1, \ldots, x_n \to h_n$ extends to a G-homomorphism ϕ: $F_H(X) \to H$. Hence, $\phi(h)$ satisfies every equation of the system $T(X) = 1$; i.e., $V_H(T) = H^n$. This shows that $F_H(X)$ is the coordinate group of H^n. Observe also that $\mathrm{Rad}(T) = \mathrm{Rad}(1)$, so $F_H(X) = G[X]/\mathrm{Rad}(1)$. To finish the proof it suffices to notice that by Proposition 2 the group $F_H(X)$ is also a free group in the variety $\mathrm{var}(H)$.

Now, we will try to describe in more detail the free group $F_{H,G}(X) = G[X]/\mathrm{Rad}_{H,G}(1)$ in $\mathrm{var}_G(H)$ with the basis X. We added temporarily the subscript G to our notations to emphasize that we consider the groups in the language L_G. In Section 2.2 we gave a formal description of $\mathrm{Rad}_{H,G}(1)$ as the set of all logical consequences of 1 with respect to the set $I_G(H)$ of all G-identities in the language L_G which hold in H. If $G = 1$ then $I_1(H)$ is the set of all identities of the standard language L (without constants from G) which hold in H. For an arbitrary group K the radical $\mathrm{Rad}_{H,1}(1)$ in K of the set $\{1\}$ with respect to $\mathrm{var}(H)$ is precisely the $I_1(H)$-verbal subgroup $\mathrm{Ver}_H(K)$ of K. The free groups $F_{H,1}(X) = F(X)/\mathrm{Ver}_H(F(X))$ in the variety $\text{var}_1(H)$ have been extensively studied (see for example [Neum]), whereas for $G \neq 1$ the structure of the G-free groups $F_{H,G}(X)$ is mostly unknown. Now, the idea is to compare the group $F_{H,G}(X)$ with its known counterpart $F_{H,1}(X)$. The following question arises naturally:

(R) for which groups H does the following G-isomorphism hold

$$F_{H,G}(X) \simeq G *_H F_{H,1}(X)?$$

Here $*_H$ is the free product in the variety $\text{var}_1(H)$ (see [Neum]).

This question has a positive answer for any torsion-free non-Abelian hyperbolic group H. Indeed, in this case H contains a free non-Abelian subgroup. Therefore $I(H)$ does not contain any non-trivial identities; hence $\text{Ver}_{H,1}(F(X)) = 1$ and $F_{H,1}(X) = F(X)$. The group $G[X] = G * F(X)$ is G-discriminated by G [BMR1], as well as by H (since G is a subgroup of H). It follows that $G * F(X) \in \text{pvar}_G(H) \subseteq \text{var}_G(H)$. Hence $\text{Rad}_{H,G}(1) = 1$ and $F_{H,G}(X) = G * F(X) = F_{H,1}(X)$, as desired.

If H is an Abelian group then the answer to the question (R) is also positive (see [Am1]), and in this case

$$F_{H,G}(X) \simeq G \times F_{H,1}(X).$$

Moreover, the result above can be generalized to groups H which are free in a nilpotent variety of groups, provided that the rank of H is not less then the nilpotency class of H [AR].

6.2. *Coordinate Groups of Algebraic Sets Over an Abelian Group*

In this section we describe coordinate groups over a given Abelian group A. Throughout this section all groups are considered in the language L_A, in which we use the additive notation for group operations. By Σ we denote the following system of quasi-identities with constants from A:

1. $\forall x \forall y (x + y = y + x)$;
2. $\forall x (nx = 0) \quad (n \in N)$;
3. $\forall x (p^n x = 0 \to p^{n-1} x = 0) \quad (n \in N,\ p \text{ prime})$;
4. $\forall x (nx = a \to x = a) \quad (n \in N, a \in A$, and $nx = a$ has no solutions in $A)$.

By Σ_A we denote the subsystem of quasi-identities from Σ which hold in A. The lemma below clarifies the meaning of the axioms Σ_A. Here we just want to mention that axiom (4) guarantees that the subgroup A is a pure subgroup in the ambient A-group. To prove the lemma we need a few notations.

The *period* of an Abelian group A is the minimal positive integer m, if it exists, such that $mA = 0$, and ∞ otherwise.

Let $T(A)$ be the torsion part of A and let $T(A) \simeq \oplus_p A_p$ be the primary decomposition of $T(A)$ (here and below p is a prime number). Denote by $e(A)$ the period of A and by $e_p(A)$ the period of $T_p(A)$. The following result describes finitely generated A-groups from the quasivariety defined by Σ_A.

LEMMA 9. *Let B be a finitely generated A-group. Then B satisfies the axioms Σ_A if and only if the following conditions hold*:

(1) $B \simeq A \oplus C$, *where C is a finitely generated Abelian group*;

(2) $e(A) = e(B)$ *and* $e_p(A) = e_p(B)$ *for every prime number* p.

Proof. Assume, first, that B satisfies the axioms Σ_A. Then A is a pure subgroup of B. Indeed, if an equation $nx = a$ ($n \in N$, $a \in A$) does not hold in A, then the axiom

$$\forall x(nx = a \to x = a)$$

belongs to the set Σ_A; hence it holds in B, which implies that the equation $nx = a$ does not have solutions in B. It follows that A is a pure subgroup in B. Observe also that B/A is finitely generated (since B is a finitely generated A-group). In this event, the subgroup A is a direct summond of B [Kap], so $B \simeq A \oplus C$, where $C \simeq B/A$. This shows that condition (1) holds in B.

Now we prove that A and B, as well as their p-torsion parts, have precisely the same periods. Let $e(A) = n$ be finite. Then the axiom $nx = 0$ is in Σ_A; hence it holds in B, and B has a period dividing n. Since A is a subgroup of B then the period of B is exactly n. In the case when $e(A) = \infty$ the group A has infinite period as well as the supergroup B. This shows that $e(A) = e(B)$. Suppose now that for a given prime p we have $e_p(A) = p^m$. Then all the axioms from Σ of the type

$$\forall x(p^n x = 0 \to p^{n-1}x = 0) \qquad (n \in N, a \in A)$$

hold in A provided $n > m$. Thus these axioms belong to Σ_A and, therefore, they hold in B. This shows that B_p has a finite period dividing p^m. Since A_p is a subgroup of B_p the period of B_p is precisely p^m. In the event when A_p has infinite period, so does the supergroup B_p. It follows that $e_p(A) = e_p(B)$, and the result holds.

Assume now that $B = A \oplus C$ where C is a finitely generated Abelian group and B has the same invariants as A. One can easily see from the argument above that the group B satisfies the axioms Σ_A. This proves the lemma.

The following theorem describes quasivarieties of Abelian groups in the language L_A.

THEOREM 1. *Let A be an Abelian group. Then an A-group B satisfies the set of axioms Σ_A if and only if $B \in$ qvar(A).*

Proof. Let $\mathcal{Q}$ be the quasivariety of A-groups defined by Σ_A. Obviously qvar$(A) \subseteq \mathcal{Q}$. We need to prove the reverse inclusion. By Lemma 1 it suffices to prove that every finitely generated A-group from $\mathcal{Q}$ belongs to qvar(A). Lemma 9 describes all finitely generated A-groups from $\mathcal{Q}$. Now to finish the proof we need the following lemma.

LEMMA 10. *Let A be an Abelian group and let B satisfy the following conditions*:

(1) $B \simeq A \oplus C$ *where C is a finitely generated Abelian group*;

(2) $e(A) = e(B)$ *and* $e_p(A) = e_p(B)$ *for every prime number p.*

Then B belongs to qvar(A).

Proof. Let

$$C \simeq \langle c_1 \rangle \oplus \cdots \oplus \langle c_t \rangle$$

be a cyclic decomposition of C into infinite cyclic groups and primary cyclic groups. Fix an arbitrary $i \in \{1, \ldots, t\}$. If c_i is an element of infinite order, then B, and hence A, has infinite period, so either there exists an element of infinite order in A or there exists an infinite sequence of elements in A with strictly increasing orders. In the latter case this sequence defines an element of infinite order in the unrestricted Cartesian power of A. In any case, qvar(A) contains an infinite cyclic group. Suppose now that c_i has a finite order, say p^n, for some prime p and positive integer n. In this event $e_p(A) = e_p(B) \geq p^n$; therefore, A has an element of order p^n. Thus, qvar(A) contains a cyclic group isomorphic to $\langle c_i \rangle$. The argument above shows that all the cyclic groups $\langle c_i \rangle$, $i = 1, \ldots, t$ belong to qvar(A). Hence the group $B \simeq A \oplus C$ belongs to qvar(A), as desired. This proves the lemma and the theorem.

Combining Proposition A, Theorem B1, Theorem 1, and Lemma 9 we obtain the following description of coordinate groups of algebraic sets over an Abelian group A. This result has been proven earlier by Fedoseeva [Fed] with the use of detailed (and rather lengthy) computations.

THEOREM D2. *Let A be an Abelian group. Then a finitely generated A-group B is a coordinate group of some algebraic set over A if and only if B satisfies the conditions*:

(1) $B \simeq A \oplus C$, *where C is a finitely generated Abelian group*;

(2) $e(A) = e(B)$ *and* $e_p(A) = e_p(B)$ *for every prime number p.*

Theorem D2 gives one a method to determine whether a given subgroup of a free group $F_A(X)$ of finite rank in the quasivariety qvar(A) is a radical subgroup or not. Namely, a subgroup I of $F_A(X)$ is radical if and only if the quotient group $B = F_A(X)/I$ satisfies conditions (1) and (2) from Lemma 9. For example, if both A and I are finitely generated, then one can effectively verify whether I is a radical subgroup of $F_A(X)$ or not.

In view of Theorem D2 we can easily describe (up to isomorphism) algebraic sets over an Abelian group A. For a coordinate group $B = A \oplus C$ over A we define *the canonical system* of equations $S = 0$ such that B is A-isomorphic to the coordinate group of the algebraic set $V_A(S)$. Fix a primary cyclic decomposition of the group C:

$$C \simeq \langle a_1 \rangle \oplus \cdots \oplus \langle a_r \rangle \oplus \langle b_1 \rangle \oplus \cdots \langle b_t \rangle.$$

Here a_i's are generators of infinite cyclic groups and b_j's are generators of finite cyclic groups of orders $p_j^{m_j}$. Let $S(X, Y) = 0$ be the following system in variables $x_1, \ldots, x_r, y_1, \ldots, y_t$:

$$p_j^{m_j} y_j = 0, \qquad j = 1, \ldots, t.$$

As we mentioned in Section 6.1 the free group $F_A(X \cup Y)$ in the variety var(A) with basis $X \cup Y$ is isomorphic to the direct product

$$A \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_r \rangle \oplus \langle y_1 \rangle \oplus \cdots \langle y_t \rangle.$$

It follows that $F_A(X \cup Y)/\langle S(X, Y) \rangle \simeq B$, so $\mathrm{Rad}_A(S) = \langle S(X, Y) \rangle$ and B is the coordinate group of the system $S(X, Y) = 0$. Clearly,

$$V_A(S) = A^r \bigoplus_{j=1}^{t} A[p_j^{m_j}],$$

where by $A[n]$ we denote the subgroup of all elements from A of orders dividing n. This proves the the following result.

PROPOSITION 9. *Let A be an Abelian group. Then any algebraic set over A is equivalent to a set of the type*

$$A^r \bigoplus_{j=1}^{t} A[p_j^{m_j}].$$

Remark 3. There is another way to describe algebraic sets over Abelian groups. Let A be an Abelian group and let $B = A \otimes C$ be a coordinate group over A. Then the algebraic set over A defined by B can be described by the set $Hom(C, A)$ of all homomorphisms from C into A. This set forms an Abelian group which can be easily recovered from A and C.

From the description of algebraic sets over an Abelian group one can immediately get the following curious result.

EXAMPLE 3. Let A be an Abelian group and $a, b \in A$. Then the set $\{a, b\}$ is an algebraic set over A if and only if there exists only one element of order 2 in A and the set $\{a, b\}$ is a coset of the unique cyclic subgroup of A of order 2.

7. COORDINATE GROUPS OF IRREDUCIBLE ALGEBRAIC SETS AND UNIVERSAL CLASSES

7.1. *Abstract Characterization of Irreducible Coordinate Groups*

In this section we characterize the coordinate groups of irreducible algebraic sets. For simplicity we call such groups *irreducible* coordinate groups.

THEOREM E1. *Let H be a G-equationally Noetherian group. Then a finitely generated G-group K is the coordinate group of an irreducible algebraic set over H if and only if* $\mathrm{ucl}(K) = \mathrm{ucl}(H)$; *i.e., K is G-universally equivalent to H.*

Proof. Let K be a finitely generated G-group G-universally equivalent to H. By Corollary 5 from Section 3.3 K is G-discriminated by H. Hence K belongs to $\mathrm{pvar}(H)$, and by Proposition A K is a coordinate group of some algebraic set $Y \subseteq H^n$. We need to prove that Y is irreducible in the Zariski topology over H^n. Suppose, to the contrary, that Y is not irreducible. Since H is G-equationally Noetherian then the Zariski topology over H^n is Noetherian. Therefore Y is a finite union of proper algebraic sets

$$Y = Y_1 \cup \cdots \cup Y_n \qquad (n \geq 2),$$

where each Y_i is defined by a system of equations, say $S_i(X) = 1$, and Y is defined by $S(X) = 1$. We know that $K \simeq G[X]/\mathrm{Rad}_H(S)$. Observe also that

$$\mathrm{Rad}_H(S) = \mathrm{Rad}_H(S_1) \cap \cdots \cap \mathrm{Rad}_H(S_n)$$

and for each i the inclusion $\mathrm{Rad}_H(S) \subsetneq \mathrm{Rad}_H(S_i)$ is proper. Let $r_i \in \mathrm{Rad}_H(S_i) - \mathrm{Rad}_H(S)$. Since K is G-discriminated by H there exists a G-homomorphism ϕ: $K \to H$ such that $\phi(r_i) \neq 1$ for each i. Notice that $\phi(X) \in V_H(S) = Y$, which implies that $\phi(X) \in Y_{i_0}$, for some i_0. It follows that $\phi(r_{i_0}) = r_{i_0}(\phi(X)) = 1$—a contradiction with the choice of ϕ. This shows that Y is irreducible.

Suppose now that a finitely generated G-group K is a coordinate group of some irreducible algebraic set $Y \subseteq H^n$ defined by a system $S(X) = 1$;

i.e., $K \simeq G[X]/\operatorname{Rad}_H(S)$. Then K belongs to pvar(H), so K is G-separated by H. We claim that K is G-discriminated by H. Indeed, suppose not. Then there are some non-trivial elements $f_1(X)\operatorname{Rad}_H(S), \ldots, f_m(X)\operatorname{Rad}_H(S) \in K (m \geq 2)$ such that any G-homomorphism ϕ: $K \to H$ vanishes on at least one of the elements f_i. Put

$$Y_i = V_H\big(S \cup \{f_i(X)\}\big) \qquad (i = 1, \ldots, m).$$

Then

$$Y = Y_1 \cup \cdots \cup Y_m$$

and for any i we have $Y \neq Y_i$ since the radicals $\operatorname{Rad}_H(S)$ and $\operatorname{Rad}_H(S \cup \{f_i(X)\})$ are different ($f_i \notin \operatorname{Rad}_H(S)$). This implies that Y is not irreducible—contradicting the assumption above. Therefore, K is G-discriminated by H and hence by Corollary 5 K is G-universally equivalent to H. We have proven the theorem.

7.2. *Universal Closures and Axioms of Irreducible Coordinate Groups*

Let H be a G-equationally Noetherian G-group. By Theorem E1 a finitely generated G-group K is the coordinate group of an irreducible algebraic set over H if and only if K is G-universally equivalent to H, which is equivalent to ucl(H) = ucl(K). In this section we describe a simple set of axioms which defines the class ucl(H) inside the quasivariety qvar(H). Moreover, if H is a non-Abelian *CSA group* then we can specify a single axiom CT (the axiom of *commutative transitivity*) such that a non-Abelian group K from qvar(H) belongs to ucl(H) if and only if K satisfies CT. A similar result holds when H is a *G-domain* (see definitions below).

Since H is G-equationally Noetherian, every algebraic set over H is defined by a finite system of equations with coefficients from G. Let n be a positive integer and let $X = \{x_1, \ldots, x_n\}$. Let $S(X), S_1(X), \ldots, S_k(X)$ be arbitrary finite subsets of $G[X]$. Then the universal formula

$$\Delta_{S, S_1, \ldots, S_k} = \forall X\Bigg(\bigg(\bigwedge_{s\in S} s(X) = 1\bigg) \leftrightarrow \Bigg(\bigg(\bigwedge_{s_1\in S_1} s_1(X) = 1\bigg) \vee \cdots \vee \bigg(\bigwedge_{s_k\in S_k} s_k(X) = 1\bigg)\Bigg)\Bigg)$$

holds in H if and only if

$$V_H(S) = V_H(S_1) \cup \cdots \cup V_H(S_k).$$

If $k \geq 2$ and all $V_H(S_i)$'s are proper subsets of $V_H(S)$ then $V_H(S)$ is not irreducible. In this event the coordinate group $K = G[X]/\operatorname{Rad}_H(S)$ of the set $V_H(S)$ does not satisfy the formula $\Delta_{S, S_1, \ldots, S_k}$. Indeed, the canonical generators $x_1 \operatorname{Rad}_H(S), \ldots, x_n \operatorname{Rad}_H(S)$ of K satisfy the conjunction $\bigwedge_{s \in S} s(X) = 1$. But if for some i they would also satisfy the conjunction $\bigwedge_{s_i \in S_i} s_i(X) = 1$, then we would have $S_i(X^\phi) = 1$ for any G-homomorphism ϕ: $G[X]/\operatorname{Rad}_H(S) \to H$, i.e., $V_H(S) = V_H(S_i)$—contradicting the conditions on S_i.

Denote by Δ the set of all formulas of the type $\Delta_{S, S_1, \ldots, S_k}$. Let Δ_H be the subset of all formulas from Δ which hold in H and let $Q(H)$ be the set of all quasi-identities in the language L_G which hold in H. The following result is a corollary of Theorem E1, the argument above, and Lemma 1 from Section 2.1.

COROLLARY 10. *Let H be a G-equationally Noetherian G-group. Then*:

(1) *A finitely generated G-group K from* $\operatorname{qvar}(H)$ *is a coordinate group of some irreducible algebraic set over H if and only if K satisfies the set of axioms Δ_H.*

(2) *A G-group K belongs to* $\operatorname{ucl}(H)$ *if and only if K satisfies the set of axioms $Q(H) \cup \Delta_H$.*

In many cases the set Δ_H above can be replaced by a much simpler set of axioms to get an axiomatization of $\operatorname{ucl}(H)$. Recall that a group H is called a *CSA-group* if every maximal Abelian subgroup M of H is malnormal; i.e., $g^{-1}Mg \cap M = 1$ for any $g \in H - M$.

We have mentioned in the Introduction that every CSA group is *commutative transitive*; i.e., it satisfies the following axiom:

$$CT = \forall x\, \forall y\, \forall z \big(x \neq 1 \wedge y \neq 1 \wedge z \neq 1 \wedge [x, y] = 1 \wedge [x, z] = 1 \to [y, z] = 1\big).$$

THEOREM E2. *Let H be a non-Abelian G-equationally Noetherian CSA G-group. Then*:

(1) *A non-Abelian finitely generated G-group K from* $\operatorname{qvar}(H)$ *is a coordinate group of some irreducible algebraic set over H if and only if K is CSA.*

(2) *A non-Abelian G-group K belongs to* $\operatorname{ucl}(H)$ *if and only if K satisfies the set of axioms $Q(H) \cup \{CT\}$ (here $Q(H)$ is the set of all quasi-identities which hold on H).*

Proof. Notice first that every non-Abelian commutative–transitive G-group K which is G-separated by the CSA group H is also CSA. Moreover, in this event K is also G-discriminated by H [BMR2]. Now, (1) follows from Theorem E1, and (2) follows from (1) and Corollary 5 from Section 3.3.

If the group G is non-Abelian then CSA G-groups form a proper subclass of a much bigger class of so-called G-domains. An element $u \in H$ is called a *G-zero divisor*, if there exists a non-trivial element $v \in H$ such that $[u, v^g] = 1$ for every $g \in G$. A group H is a *G-domain* if it does not have any G-zero divisors. Clearly, a G-group H is a G-domain, if and only if the following infinite formula holds in H:

$$\mathrm{Dom}_{G,\infty} = \forall x\, \forall y\Big(\bigwedge_{g\in G} [x, y^g] = 1 \to x = 1 \vee y = 1\Big).$$

If H is G-equationally Noetherian then the system $\bigwedge_{g\in G}[x, y^g] = 1$ is equivalent over H to a finite subsystem. Hence the formula $\mathrm{Dom}_{G,\infty}$ is equivalent over H to a universal formula of the following type:

$$\mathrm{Dom}_H = \forall x\, \forall y\big([x, y^{g_1}] = 1 \wedge \cdots \wedge [x, y^{g_k}] = 1 \to x = 1 \vee y = 1\big).$$

Observe that the system $\bigwedge_{g\in G}[x, y^g] = 1$ is equivalent to the same finite subsystem over an arbitrary group from $\mathrm{qvar}(H)$. This allows one to prove the following result (similar to Theorem E2).

PROPOSITION 10. *Let H be a G-equationally Noetherian G-domain. Then*:

1. *A finitely generated G-group K from* $\mathrm{qvar}(H)$ *is the coordinate group of an irreducible algebraic set over H if and only if K is a G-domain.*

2. *A G-group K belongs to* $\mathrm{ucl}(H)$ *if and only if K satisfies the set of axioms $Q(H) \cup \{\mathrm{Dom}_H\}$.*

7.3. *Abelian Case*

In the rest of this section we describe the coordinate groups of irreducible algebraic sets over a fixed Abelian group A and give the set of axioms in the language L_A for the universal closure $\mathrm{ucl}(A)$ of A. Moreover, for an algebraic set over A we describe its irreducible components.

For a positive integer k and a prime number p we denote by $\alpha_{p^k}(A)$ the number of elements of order p^k in A (we allow $\alpha_{p^k}(A) = \infty$).

THEOREM E3. *Let A be an Abelian group and let B be a finitely generated A-group. Then B is a coordinate group of some irreducible algebraic set over A if and only if the following conditions hold*:

(1) *$B \simeq A \oplus C$ for some finitely generated Abelian group C*;

(2) *$e(A) = e(B)$ and $e_p(A) = e_p(B)$ for each prime number p*;

(3) *$\alpha_{p^k}(A) = \alpha_{p^k}(B)$ for each prime number p and positive integer k.*

Proof. By Theorem D2 a finitely generated A-group B is a coordinate group over A if and only if B satisfies the conditions (1) and (2).

Suppose now that (3) does not hold in B for some particular p and k. This implies that $\alpha_{p^k}(A)$ is finite and $\alpha_{p^k}(A) < \alpha_{p^k}(B)$; i.e., A has fewer elements of order p^k then B. It follows then that B is not A-discriminated by A. Hence B is not a coordinate group of an irreducible algebraic set over A. This proves that if B is a coordinate group of an irreducible set over A then conditions (1), (2), (3) hold in B.

Let us assume that B satisfies conditions (1), (2), (3). Then $B = A \oplus C$ for some finitely generated Abelian group C. Let

$$C \simeq Z^t \oplus_p T_p(C) \tag{6}$$

be a decomposition of of C into a torsion-free part Z^t and the direct sum of p-components of the torsion part $T(C)$. We want to show that B is A-discriminated by A. It suffices to show that for any finite subset of non-trivial elements $A_0 \subseteq A$ and any finite subset of non-trivial elements $D_0 \in Z^t$ there exists an A-homomorphism $B \to A$ which does not vanish on every non-trivial element from the finite set

$$A_0 + D_0 + T(C).$$

The proof is divided into two cases.

(1) We show, first, how to discriminate $A_0 + D_0$ into A by an A-homomorphism ϕ: $A \oplus Z^t \to A$. If $t = 0$ there is nothing to prove. Suppose $t \neq 0$. It is easy to see that Z^t is discriminated by Z (see [BMR2]); hence it suffices to consider the case when $t = 1$, i.e., $D_0 \subseteq Z$. We may assume now that Z is the additive group of the ring of integers, so every element in Z is just an integer. This allows one for a given fixed element $b \in A$ to define an A-homomorphism ϕ: $A \oplus Z \to A$ as follows:

$$\phi\colon (a + z) \to (a + zb) \qquad a \in A, \qquad z \in Z,$$

Notice that since $t \neq 0$ the period of A is infinite. Therefore, for every integer n there exists an element in A of order greater then n. We claim that there exists an element $b \in A$ of a sufficiently big order (or infinite order) such that the images of all non-trivial elements from $A_0 + D_0$ under the homomorphism ϕ, defined above, are non-trivial. To see this, decompose the finitely generated subgroup $\langle A_0 \rangle$ of A into a torsion-free part Z^s and the torsion part $T(\langle A_0 \rangle)$. Then each element x from $A_0 + D_0$ can be written uniquely in the form

$$x = a_x + c_x + d_x, \qquad a_x \in T(\langle A_0 \rangle), \qquad c_x \in Z^s, \qquad d_x \in D_0.$$

If $s \neq 0$ choose $b \in Z^s$ such that $c_x \neq d_x b$ for any non-trivial $x \in A_0 + D_0$. If $s = 0$ choose b such that each $d_x b$ has the order bigger then the order of $T(\langle A_0\rangle)$. In any event $\phi(x) \neq 1$ for every non-trivial $x \in A_0 + D_0$, as claimed.

(2) We show now how one can discriminate $A_0 + D_0 + T(C)$ into A. Let $p_1, \ldots, p_e$ be all prime numbers for which $T_{p_i}(C)$ is not trivial. Take an arbitrary $p = p_i$. Observe that for every positive integer k we have

$$\alpha_{p^k}(B) = \alpha_{p^k}(A) + \alpha_{p^k}(T_p(C)).$$

Therefore, if $\alpha_{p^k}(A)$ is finite then $\alpha_{p^k}(T_p(C)) = 0$, which implies that C does not contain elements of order p^k. Let p^k be the maximal order among all the elements from $T_p(C)$. In this event, $\alpha_{p^k}(A)$ is infinite; hence A contains an infinite direct sum $\bigoplus_{i\in I} Z_{p^k}$ of copies of a cyclic group Z_{p^k} of order p^k. This implies that for any infinite subset of indexes $J \subseteq I$ the subgroup $\bigoplus_{i\in J} Z_{p^k}$ contains a subgroup isomorphic to $T_p(C)$ (which is a finite direct sum of finite cyclic p-groups of order $\leq p^k$). For a finite subset $X \subseteq A$ there exists a finite set of indexes $I_X \subseteq I$ such that the intersection of the subgroup generated by X and $\bigoplus_{i\in I} Z_{p^k}$ is contained in $\bigoplus_{i\in I_X} Z_{p^k}$. Hence the subset $J_X = I - I_X$ is infinite and $T_p(C)$ is embeddable into $\bigoplus_{i\in J_X} Z_{p^k}$. Now, put $X_0 = \phi(A_0 + D_0)$ and denote by ϕ_1 an embedding

$$\phi_1\colon T_{p_1}(C) \to \bigoplus_{i\in J_{X_0}} Z_{p^k}.$$

It follows that the homomorphism

$$\psi_1 = \phi \oplus \phi_1\colon A \oplus Z \oplus T_{p_1}(C) \to A$$

discriminates the set $A_0 + D_0 + T_{p_1}(C)$ into A. Put

$$X_1 = \psi_1(A_0 + D_0 + T_{p_1}(C))$$

and proceed as above. By induction we construct an A-homomorphism

$$\psi_e\colon A \oplus Z \bigoplus_{i=1}^{e} T_{p_i}(C) \to A$$

that discriminates all non-trivial elements from $A_0 + D_0 + T(C)$. This proves the theorem.

If $\alpha_{p^k}(A) = m$ is finite and $\{a_1, \ldots, a_m\}$ is the set of all elements of order p^k in A then A satisfies the following universal formula with constants from A:

$$\psi_{p^k} = \forall y\big(p^k y = 0 \wedge p^{k-1}y \neq 0 \to (y = a_1 \vee \cdots \vee y = a_m)\big).$$

If the formula ψ_{p^k} also holds in an A-group B then $\alpha_{p^k}(A) = m = \alpha_{p^k}(B)$. Denote by Ψ_A the set of all formulas of the type ψ_{p^k} (for all p and k) which hold in A.

THEOREM 2. *Let A be an Abelian group. Then an A-group B belongs to the universal closure* $\mathrm{ucl}(A)$ *of A if and only if B satisfies the set of axioms* $\Sigma_A \cup \Psi_A$.

Proof. Clearly, every group from $\mathrm{ucl}(A)$ satisfies the axioms $\Sigma_A \cup \Psi_A$. To prove the reverse inclusion consider an A-group B which satisfies the set of axioms $\Sigma_A \cup \Psi_A$. If B is finitely generated as an A-group then by Theorem D2, Theorem E3, and the argument above B is the coordinate groups of an irreducible algebraic set over A. Since A is A-equationally Noetherian it follows from Theorem E1 that B belongs to $\mathrm{ucl}(A)$. If B is not finitely generated as an A-group then every one of its finitely generated A-subgroup $B_0 \le B$ satisfies the axioms $\Sigma_A \cup \Psi_A$. It follows by the argument above that $B_0 \in \mathrm{ucl}(A)$, i.e., the group B is locally discriminated by A. Now by Theorem B2 $B \in \mathrm{ucl}(A)$, which proves the result.

COROLLARY 11. *If A is a torsion-free Abelian group, then* $\mathrm{ucl}(G) = \mathrm{qvar}(G)$.

Now for an Abelian group A we describe the irreducible components of the algebraic set Y over A.

We start by defining new invariants of an Abelian group A. For a prime number p denote by $\gamma_p(A)$ the maximal integer k for which $\alpha_{p^k}(A) = \infty$, if a such k exists, and $\gamma_p(A) = 0$ otherwise. For a positive integer m denote by $A[m]$ the subgroup of all elements in A the orders of which divide m.

PROPOSITION 11. *Let $B = A \oplus C$ be a coordinate group over A and assume that Z^t is a torsion-free part of C. Then*

$$B_{\mathrm{irr}} = A \oplus Z^t \oplus_p C[p^{\gamma_p(A)}]$$

is the maximal subgroup of B which is an irreducible coordinate group over A.

Proof. Let D be an A-subgroup of B which is an irreducible coordinate group over A. Then by Theorem 1 $D \simeq A \oplus E$, where E is a finitely generated Abelian group, and $e(A) = e(D)$, $e_p(A) = e_p(D)$, $\alpha_{p^k}(A) = \alpha_{p^k}(D)$ for every prime p and positive integer k. Observe that if the torsion-free part Z^t of C is non-trivial then $e(A) = \infty$ and the subgroup $\langle D, Z^t \rangle$ generated by D and Z^t has precisely the same invariants as A. So B_{irr} has to contain Z^t. Notice that if $\alpha_{p^k}(A)$ is finite then $\alpha_{p^k}(D) = 0$. And if $\alpha_{p^k}(A) = \infty$ then B_{irr} has to contain all the elements of order p^k from C. It follows that the maximal A-subgroup of B which is an irreducible coordinate group over A has to contain

$$A \oplus Z^t \oplus_p C[p^{\gamma_p(A)}].$$

Now it is suffices to check that this subgroup indeed has the same invariants as A.

THEOREM 3. Let A be an Abelian group and let B be a coordinate group of an algebraic set V_B over A. Then all irreducible components of V_B are isomorphic, as algebraic sets, to the set $V_{B_{\text{irr}}}$ defined by B_{irr}. Moreover, the number of irreducible components of V_B is equal to the index of B_{irr} in B.

Proof. We use notations introduced above. Let $B = A \oplus C$. Fix a primary cyclic decomposition of the group C:

$$C \simeq \langle x_1 \rangle \oplus \cdots \oplus \langle x_r \rangle \oplus \langle y_1 \rangle \oplus \cdots \langle y_t \rangle /$$

Here x_i's are generators of infinite cyclic groups and y_j's are generators of finite cyclic groups of orders $p_j^{m_j}$. Let $S(X, Y) = 0$ be the following system in variables $x_1, \ldots, x_r, y_1, \ldots, y_t$:

$$p_j^{m_j} y_j = 0, \qquad j = 1, \ldots, t.$$

Clearly,

$$V_A(S) = A^r \bigoplus_{j=1}^{t} A[p_j^{m_j}],$$

and the coordinate group over A defined by S is A-isomorphic to B. Let $\beta_j = \min\{m_j, \gamma_{p_j}(A)\}$, $j = 1, \ldots, t$. Define the system $S_{\text{irr}}(X, Y) = 0$ as follows:

$$p_j^{\beta_j} y_j = 0, \qquad j = 1, \ldots, t.$$

Obviously, this system defines the algebraic subset $V_A(S_{\text{irr}})$ of $V_A(S)$ with the coordinate group isomorphic to B_{irr}. Hence the algebraic set $V_A(S_{\text{irr}})$ is irreducible. Observe that both the sets $V_A(S_{\text{irr}})$ and $V_A(S)$ are subgroups of A^{r+t} and the index of $V_A(S_{\text{irr}})$ in $V_A(S)$ is finite and equal to the index of B_{irr} in B. It follows that $V_A(S)$ is a union of finitely many cosets of $V_A(S_{\text{irr}})$, each of which is irreducible algebraic set (since all of them are isomorphic to $V_A(S_{\text{irr}})$). Hence these cosets are irreducible components of $V_A(S)$, as desired.

8. CONNECTED COMPONENTS

In the classical algebraic groups the notion of connected components plays an important part. In this section we study an analog of this notion for Zariski topology over equationally Noetherian groups. Let G be an equationally Noetherian group. The following question naturally arises here: what are the irreducible components of G? Or, in more general setting: what are irreducible components of "algebraic groups" over G? It turns out that, similar to the classical case, the connected component H_0 of an algebraic group H over G is a normal subgroup of finite index and all irreducible components of H are cosets of H_0.

Now we define an algebraic group over a group G.

DEFINITION 5. Let $V \subseteq G^n$ be an algebraic set over G. Assume that V forms a group H with respect to some operations of multiplication $\circ$ and inversion $^{-1}$. The group $H = \langle V; \circ, ^{-1} \rangle$ is called *algebraic over* G if the operations $\circ$ and $^{-1}$ can be defined by word mappings with coefficients in G.

The group H is called *a standard algebraic group over* G if $\circ$ and $^{-1}$ are restrictions of the standard multiplication and inversion on G^n.

LEMMA 11. *Let* $H = \langle V; \circ, ^{-1} \rangle$ *be an algebraic group over* G. *Then*

(1) *the Zariski topology on* H *is weaker then the topology induced on* H *by the Zariski topology on* G^n;

(2) *if* G *is equationally Noetherian, then* H *is equationally Noetherian.*

Let $H = \langle V; \circ, ^{-1} \rangle$ be an algebraic group over G. A subgroup $K \leq H$ is called *an algebraic subgroup over* G if K is an algebraic set over G. The following is a corollary of Lemma 11.

COROLLARY 12. *Let* $H = \langle V; \circ, ^{-1} \rangle$ *be an algebraic group over* G. *Then a subgroup* $K \leq H$ *which is algebraic over* H *is also algebraic over* G.

LEMMA 12. *Let* G *be an equationally Noetherian group and let* H *be an algebraic group over* G. *Then there exists a unique minimal algebraic over* G *subgroup* H_0 *of* H *of finite index. This subgroup* H_0 *is normal in* H.

Proof. Observe that the intersection of two algebraic subgroups of finite index in H is again an algebraic subgroup of finite index in H. Since G is equationally Noetherian it satisfies the descending chain condition on closed subsets. It follows that the intersection H_0 of all algebraic over G subgroups of H of finite index is equal to the intersection of finitely many such subgroups. Hence H_0 is the minimal algebraic over G subgroup of H of finite index in H. To see that H_0 is normal it suffices to notice that if N is an algebraic subgroup of H of finite index, then for any $h \in H$ the subgroup $h^{-1}Nh$ is also an algebraic subgroup of H of finite index. This proves the lemma.

DEFINITION 6. Let G be an equationally Noetherian group and let H be an algebraic group over G. Then the unique minimal algebraic over G subgroup of H of finite index is called the connected component of H. We denote it by H_0.

THEOREM 4. *Let G be an equationally Noetherian group and let $H = \langle V; \circ, ^{-1} \rangle$ be an algebraic group over G. Then the connected component H_0 of H is an irreducible component of V and all other irreducible components of V are cosets of H_0 in H.*

Proof. Let $H = \langle V; \circ, ^{-1} \rangle$ be an algebraic group over G. Since G is equationally Noetherian V admits decomposition into finitely many irreducible components

$$V = V_1 \cup \cdots V_n.$$

For any $h \in H$ the sets $h \circ V_i$ and $V_i \circ h$ are also algebraic over G and

$$V = V_1 \circ h \cup \cdots \cup V_n \circ h = h \circ V_1 \cup \cdots \cup h \circ V_n.$$

This shows that $V_i \circ h$ and $h \circ V_i$ are also irreducible components of V. Hence H acts by right (as well as left) multiplication on the set $\{V_1, \dots, V_n\}$ of irreducible components of V. Suppose that the identity element 1 of H (with respect to the multiplication $\circ$) belongs to V_1. Put

$$K = \{h \in H \mid V_1 \circ h = V_1\}.$$

Obviously, K is a normal subgroup of finite index in H. Since $1 \in V_1$, it follows that $K \le V_1$. We claim that K is an algebraic set over G. Indeed, assume that V_1 is defined over G by a system of equations $S_1(y_1, \dots, y_n) = 1$ with coefficients from G. Then K is defined by the system

$$s(g \circ y) = 1, \qquad s \in S_1, \qquad g \in V_1, \qquad y = (y_1, \dots, y_n).$$

The system above is equivalent to a finite part of it, since G is equationally Noetherian. Since K is of finite index in H we have

$$V_1 = K \circ h_1 \cup \cdots \cup K \circ h_n$$

for some $h_1, \dots, h_n \in H$. This implies (in view of irreducibility of V_1) that $V_1 = K$. It follows now that

$$V = K \circ g_1 \cup \cdots \cup K \circ g_n$$

for some elements $g_1, \dots, g_n \in H$. Hence the cosets of K are precisely the irreducible components of V, as desired.

Recall that a topological space is *connected* if it is not a disjoint union of two non-empty closed subsets. Observe that every algebraic set over G irreducible in the Zariski topology is connected. Theorem 4 shows that if G is equationally Noetherian then an algebraic over G group $H = \langle V; \circ, ^{-1} \rangle$ is connected (in the topology induced from G^n) if and only if V is irreducible, which is equivalent to $H = H_0$.

It is a well known result in topology that if topological spaces X and Y are connected then their product $X \times Y$ is also connected (in the product topology). If H and K are algebraic groups over G then their direct product $H \times K$ is also algebraic over G (since a product of algebraic sets over G is again algebraic; see [BMR1]). Notice, that the Zariski topology on $H \times K$ might not be the product topology of H and K.

Remark 4. Let F be a free non-Abelian group. Then the Zariski topology on $F \times F$ is not the product topology of the Zariski topology on F.

Nevertheless, the following result holds for the Zariski topologies.

LEMMA 13. *Let G be a group. If $V \subseteq G^n$ and $W \subseteq G^m$ are connected algebraic sets over G then their product $V \times W$ is a connected algebraic set over G.*

Proof. It is not hard to see that the direct product of two algebraic sets V, W over G is an algebraic set over G [BMR1]. So it is suffices to show that if both V, W are connected then $V \times W$ is connected. It is readily seen that for any $w \in W$ the subset $V \times \{w\} \subseteq V \times W$ in the topology induced from G^{m+n} is homeomorphic to V with respect to the map $v \to (v, w)$, where $v \in V$. Thus for any $v \in V, w \in W$ the sets $V \times \{w\}$, $\{v\} \times W$ are connected. Since they both contain (v, w) their union $V \times \{w\} \cup \{v\} \times W$ is connected. Notice that for a fixed $w \in W$ one has

$$V \times W = \bigcup_{v \in V} \big(V \times \{w\} \cup \{v\} \times W\big).$$

Again, since

$$\bigcap_{v \in V} \big(V \times \{w\} \cup \{v\} \times W\big) \neq \emptyset$$

the union above is connected, as needed.

We say that an algebraic group $H = \langle V; \circ, ^{-1} \rangle$ over G is *connected* if the set V is connected in the Zariski topology over G.

PROPOSITION 12. *Let G be a group and let H, K be algebraic groups over G. Then $H \times K$ is connected if and only if the groups H and K are connected.*

Proof. Let $H = \langle V; \circ, ^{-1} \rangle$ and $K = \langle W; \circ, ^{-1} \rangle$ be algebraic groups over G with connected sets V and W. Then by Lemma 13 the product $V \times W$ is connected; i.e., the group $H \times K$ is connected. The reverse is also true since the canonical projections $V \times W \to V$ and $V \times W \to W$ are continuous in the Zariski topology and the image of a connected set is connected under a continuous map.

Summarizing the discussion above we have the following result.

THEOREM 5. *Let G be an equationally Noetherian group. If H and K are algebraic groups over G then* $(H \times K)_0 = H_0 \times K_0$.

Proof. By Proposition 12 the group $H_0 \times K_0$ is connected. Moreover, $H_0 \times K_0$ has finite index in $H \times K$. Hence

$$(H \times K)_0 \leq H_0 \times K_0.$$

By Theorem 4 $(H \times K)_0$ has finite index in $H \times K$, as well as in $H_0 \times K_0$. But the group $H_0 \times K_0$ is connected, therefore $(H \times K)_0 = H_0 \times K_0$.

Finally we describe connected components of algebraic groups over an Abelian group A.

LEMMA 14. *Let A be an Abelian group. Then any algebraic set $V \subset A^n$ can be turned into an algebraic group over A for a suitable choice of multiplication and inversion on V.*

Proof. As we saw above every algebraic set $V \subset A^n$ is either a subgroup of A^n or a coset $W + a$ of some subgroup W of A^n. In the former case V forms an algebraic subgroup of A^n. In the latter case the group multiplication $\circ$ on $W + a$ is defined as follows:

$$(w_1 + a) \circ (w_2 + a) = w_1 + w_2 + a \qquad (w_1, w_2 \in W).$$

It is easy to see that $W + a$ forms a group (isomorphic to W) with respect $\circ$. This proves the lemma.

COROLLARY 13. *In the notations as above let $V = W + a \subset A^n$ be an algebraic set over an Abelian group A. Then there exists a subgroup W_0 of finite index in W such that the irreducible components of V are cosets of W_0.*

Proof. This follows from the lemma above and Theorem 4.

Remark 5. The corollary above gives another view of Theorem 2. The subgroup W_0 (in the notations as above) is precisely the subgroup corresponding to the algebraic set defined by the group B_{irr} from Theorem 3.

9. OPEN PROBLEMS AND QUESTIONS

We have mentioned in Section 3.2 that Abelian groups are q_ω-compact (in fact, they are equationally Noetherian). This prompts us to formulate the following problem.

Problem 1. (1) *Describe the q_ω-compact nilpotent groups of class* 2.

(2) *Describe the q-compact nilpotent groups of class* 2.

The solution of the next two problems would either strengthen the results in Section 3 or provide interesting examples of pathological groups.

Problem 2. Are there any q_ω-compact groups which are not equationally Noetherian?

Problem 3. Are there any q-compact groups which are not u-compact?

In [BMR1] we asked whether or not every hyperbolic group is equationally Noetherian and also whether or not a free product of two equationally Noetherian groups is equationally Noetherian. Now we can formulate a weaker version of these problems which might be easier to solve.

Problem 4. Is every hyperbolic group q_ω-compact?

Problem 5. Is it true that a free product of q_ω-compact groups is q_ω-compact?

It is known [BMR1] that for equationally Noetherian groups every algebraic set is a finite union of irreducible components. Whether or not the converse is true is unknown. The problem below is stated to clarify the relationship between q_ω-compactness and the decomposition of algebraic sets into irreducible components.

Problem 6. (1) *Let H be a* q_ω*-compact G-group. Is it true that every algebraic set over H is a finite union of its irreducible components*?

(2) *Is it true that if every algebraic set over H is a finite union of irreducible algebraic sets then H is* q_ω*-compact*?

Due to the classical Chevalley–Kolchin theorem every algebraic matrix group can be completely described by a set of polynomial semi-invariants.

Problem 7. Let G be a group. Does an analog of the Chevalley–Kolchin theorem hold for algebraic groups over G?

To make this problem more precise we need the following definition. Let $G[X] = G[x_1, \dots, x_n]$ be a free G-group of rank n. Assume that r is a positive integer less or equal then n. Denote by $\lambda\colon G^r \to \mathrm{Aut}_G(G[X])$ an embedding of G^r into the group of all G-automorphisms of $G[X]$. For a subset $S \subset G[X]$ the set

$$\mathrm{Stab}_\lambda(S) = \{ g = (g_1, \dots, g_r) \in G^r \mid \lambda(g)(s) = s \, \forall s \in S \}$$

is called the *λ-stabilizer* of S. Now, the problem above can be stated as follows: is it possible to describe algebraic groups over G in terms of the stabilizers $\mathrm{Stab}_\lambda(S)$?

By Theorem E2, if F is a non-Abelian free group then a set of axioms of the universal theory of a F can be obtained by adding the axiom of transitivity of commutation CT to an arbitrary set of quasi identities which axiomatizes the quasivariety qvar(F). This explains the importance of the following problem which is, probably, due to Malcev.

Problem 8. Find a "nice" set of axioms for the quasivariety qvar(F) for a free non-Abelian group F.

Problem 9. Find a "nice" set of axioms for the quasivariety qvar(G) for a free non-Abelian nilpotent (solvable) group G.

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